15-03-2022

yesterday

additional material
- Mailath p. 292 →
- Borgess p. 07 →

1) Revelation principle

↪ Allows us to look at direct, truthful mechanisms

2) used incentive constraints

(i) $q(\cdot)$ increasing

(ii) $V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} q(s)\,ds$

& $t(\theta) = V(\theta) - \theta q(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} q(s)\,ds - \theta q(\theta)$

(iii) $V(\underline{\theta}) \geq 0 \Rightarrow IR_{\theta}$ holds for all θ

3) "order of integration trick"

to get

$$\int_{\underline{\theta}}^{\bar{\theta}} \Big[\underbrace{\theta q(\theta) - \int_{\underline{\theta}}^{\theta} q(s)\,ds}_{= -t(\theta)}\Big] f(\theta)\,d\theta$$

$$= \int_{\underline{\theta}}^{\bar{\theta}} J(\theta) \cdot q(\theta) f(\theta)\,d\theta$$

with $J(\theta) = \theta - \dfrac{1-F(\theta)}{f(\theta)}$

today

1) some direct extensions

2) formal proofs of steps 2) & 3)

Screening – some immediate extensions of example

Divisible quantity instead of single indivisible good:

- nothing changes
- interpret $q \in [0, \bar{q}]$ as quantity instead of probability $q \in [0, 1]$

Production costs for the seller:

- suppose seller incurs cost $c(q)$ when producing quantity q
- seller's objective is now $-t(\theta) - c(q(\theta))$
- for buyer, nothing changes
- optimality condition for pointwise maximisation (if c convex increasing):

$q^*(\theta)$ determined by

$$\boxed{\theta - \frac{1 - F(\theta)}{f(\theta)} - c'(q(\theta)) = 0.}$$

Screening – incentive compatibility (formally)

We made our lives easy at several steps of the example:

1) did not proof formally that q must be increasing to fulfil IC
2) did not proof formally that $V'(\theta) = q(\theta)$ must hold to fulfil IC

(3) Buyer's linear utility $\theta q + t$ seems like (very simple) special case

Let's provide complete proofs of (1) and (2) for more general case when buyer's utility is

$$\boxed{u(\theta, q) + t.}$$ – "quasi-linear utility"

↳ yesterday $u(\theta, q) = \theta \cdot q$

Screening – incentive compatibility – three results

We consider three fundamental results:

1. Envelope Theorem
2. Revenue Equivalence
3. ‘Incentive Compatibility Characterisation’

paper: Milgrom & Segal (2002) Econometrica. "Envelope Theorems"

Theorem (Envelope Theorem)

Assume that X is compact, and $\Theta = [\underline{\theta}, \overline{\theta}]$ and $g : \Theta \times X \to \mathbb{R}$ is differentiable in θ with uniformly bounded derivative. * *Suppose the selection $x^*(\theta)$ solves*

$$\boxed{V(\theta) = \max_{x \in X} g(\theta, x).}$$

Then we have

$$V'(\theta) = \frac{\partial}{\partial \theta} g(\theta, x^*(\theta)) \quad \text{a.e.}$$

almost every

and

$$\boxed{V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \frac{\partial}{\partial \theta} g(s, x^*(s))\, \mathrm{d}s.}$$

- X is our space of outcomes. In our example $X = [0,1] \times \mathbb{R}$, $x = (q, t)$ | $g(\theta; (q,t)) = \theta \cdot q + t$

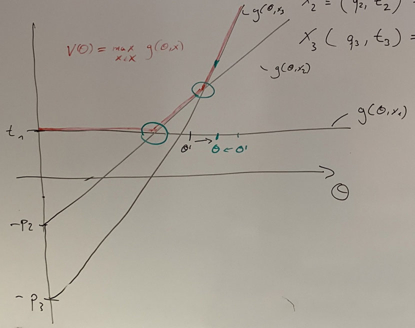
* there exists $b \in \mathbb{R}$ s.t. $\max_{x \in X} \left| \frac{\partial}{\partial \theta} g(\theta, x) \right| \le b \quad \forall \theta$.

example of $V(\theta)$ with $g(\theta, x) = g(\theta; q, t)) = \theta \cdot q + t$

suppose $X = \{x_1, x_2, x_3\}$: $x_1 = (q_1, t_1) = (0, t_1)$

$x_2 = (q_2, t_2) = (1/2, -p_2)$

$x_3 (q_3, t_3) = (1, -p_3)$



Screening – Envelope Theorem – proof

First, we show that $V(\cdot)$ is absolutely continuous so that

$V'(\cdot)$ exists a.e. and $V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} V'(s)\, ds$.

- For any θ, θ' $|V(\theta') - V(\theta)| \leq \max_{x \in X} |g(\theta', x) - g(\theta, x)| = \max_x \left| \int_{\theta}^{\theta'} \frac{\partial}{\partial \theta} g(s, x)\, ds \right|$

$$\leq \int_{\theta}^{\theta'} \underbrace{\max_x \left| \frac{\partial}{\partial \theta} g(s, x) \right|}_{\leq b} ds \leq \int_{\theta}^{\theta'} b\, ds$$

$\Rightarrow V(\cdot)$ is absolutely continous.

Second, we show that $V'(\theta) = \frac{\partial}{\partial\theta} g(\theta, x^*(\theta))$ a.e.

Take any $\theta \neq \theta'$

$V(\theta') \geq g(\theta', x^*(\theta))$ — true preference θ' but selection for other type θ

$V(\theta') \geq \underbrace{V(\theta) - g(\theta, x^*(\theta))}_{=0 \text{ by definition}} + g(\theta', x^*(\theta))$

$\Leftrightarrow V(\theta') - V(\theta) \geq g(\theta', x^*(\theta)) - g(\theta, x^*(\theta))$

case 1: $\theta' > \theta$: divide by $(\theta' - \theta) > 0$

$$\frac{V(\theta') - V(\theta)}{\theta' - \theta} \geq \frac{g(\theta', x^*(\theta)) - g(\theta, x^*(\theta))}{\theta' - \theta}$$

limit $\theta' \searrow \theta$: $V'(\theta_+) \geq \frac{\partial}{\partial\theta} g(\theta, x^*(\theta))$

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case 2: $\theta' < \theta$: divide by $(\theta' - \theta) < 0$

$$\frac{V(\theta') - V(\theta)}{\theta' - \theta} \leq \frac{g(\theta', x^*(\theta)) - g(\theta, x^*(\theta))}{\theta' - \theta}$$

limit $\theta' \nearrow \theta$: $V'(\theta_-) \leq \frac{\partial}{\partial\theta} g(\theta, x^*(\theta))$

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since $V'(\cdot)$ exists a.e. i.e. $V'(\theta) = V'(\theta_+) = V'(\theta_-) = \frac{\partial}{\partial\theta} g(\theta, x^*(\theta))$

and thus $V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \frac{\partial}{\partial\theta} g(s, x^*(s))\, ds$ $\square$

Screening – Revenue Equivalence

Theorem (Revenue Equivalence)

Fix a function $q : \theta \to Q$. Suppose that Q is compact and $\Theta = [\underline{\theta}, \overline{\theta}]$. Let the agent's utility be $u(\theta, q) + t$, where u is differentiable in θ with uniformly bounded derivative. Any incentive compatible mechanism that implements $q(\theta)$ gives agent payoff

$$V(\theta) = \boxed{V(\underline{\theta})} + \int_{\underline{\theta}}^{\theta} \frac{\partial}{\partial \theta} u(s, q(s))\, \mathrm{d}s,$$

transfers must satisfy $\boxed{-t(\theta) = u(\theta, q(\theta)) - V(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} \frac{\partial}{\partial \theta} u(s, q(s))\, \mathrm{d}s.}$

- By IC, the allocation rule almost completely pins down agent's and principal's payoff
- Only 'degree of freedom' is the constant $V(\underline{\theta})$

proof idea: use envelope Theorem with $g(\theta; (q,t)) = u(\theta, q) + t$
$(q(\theta), t(\theta))_\theta$ is incentive compatible if $x^*(\theta) = (q(\theta), t(\theta))$

Screening – characterisation of IC

The Revenue Equivalence Theorem provides a necessary condition:

'If mechanism is incentive compatible, then (q, t) *satisfies...'*

Two issues remain:

- (When) are these conditions sufficient for incentive compatibility?
- In our example we said q had to be increasing, where did that come from?

Screening – characterisation of IC

Theorem

Suppose that Q is compact and $\Theta = [\underline{\theta}, \overline{\theta}]$. Let the agent's utility be $u(\theta, q) + t$, where u is differentiable in θ with uniformly bounded derivative.

If $\frac{\partial^2 u(\theta,q)}{\partial q \partial \theta} > 0$, then $(q(\theta), t(\theta))$ is IC if and only if

with $u(\theta,q) = \theta \cdot q$,
$\frac{\partial^2}{\partial\theta\partial q} = 1 > 0.$

$$q(\theta) \text{ is non-decreasing}$$

and

$$-t(\theta) = u(\theta, q(\theta)) - V(\underline{\theta}) - \int_{\underline{\theta}}^{\theta} \frac{\partial}{\partial \theta} u(s, q(s)) \, \mathrm{d}s.$$

Screening – characterisation of IC – proof

Incentive compatibility means that for all θ and θ'

$$\underline{u(\theta, q(\theta)) + t(\theta)} - [u(\theta, q(\theta')) + t(\theta')] \geq 0$$

$$= \overbrace{V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \frac{\partial}{\partial\theta} u(s, q(s))\, ds} - [u(\theta, q(\theta')) \overbrace{- u(\theta', q(\theta')) + V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta'} \frac{\partial}{\partial\theta} u(s, q(s))\, ds}]$$

$$= \underbrace{\int_{\underline{\theta}}^{\theta} \frac{\partial}{\partial\theta} u(s, q(s))\, ds - \int_{\underline{\theta}}^{\theta'} \frac{\partial}{\partial\theta} u(s, q(s))\, ds}_{= \int_{\theta'}^{\theta} \frac{\partial}{\partial\theta} u(s, q(s))\, ds} - \underbrace{[u(\theta, q(\theta')) - u(\theta', q(\theta'))]}_{= \int_{\theta'}^{\theta} \frac{\partial}{\partial\theta} u(s, q(\theta'))\, ds}$$

$$= \int_{\theta'}^{\theta} \left(\frac{\partial}{\partial\theta} u(s, q(s)) - \frac{\partial}{\partial\theta} u(s, q(\theta')) \right) ds = \int_{\theta'}^{\theta} \int_{q(\theta')}^{q(s)} \underbrace{\frac{\partial^2}{\partial\theta\partial q} u(s, z)}_{> 0 \text{ by assumption}}\, dz\, ds$$

≥ 0 for all θ, θ' iff $(\theta - \theta')(q(\theta) - q(\theta')) \geq 0$ for all θ, θ' $\square$

Adverse Selection

Mechanism Design

Mechanism Design

Mechanism design

- how can we aggregate individual preferences into a collective decision?
- especially if individuals' preferences are private information

Compared to the screening problem, we now consider multiple agents

- interests may conflict with each other
- there is increased competition that a seller may exploit
- will inefficiencies increase/decrease?

The Environment

- n agents
- each agent i has private information (his type) $\theta_i \in \Theta_i$
- set of possible alternatives/outcomes $x \in X$
- each agent is expected-utility maximiser with vNM utility function

$$u_i(\theta, x) \in \mathbb{R}, \qquad \text{for } \theta \in \Theta = \Theta_1 \times \cdots \times \Theta_n \text{ and } x \in X.$$

- the type profile $\theta = (\theta_1, \ldots, \theta_n)$ is distributed according to F with density $f > 0$
- **notation:** we write

$$\boxed{\theta_{-i}} = (\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n) \qquad \text{and} \qquad (\theta_i, \theta_{-i}) = \theta$$

θ_i is missing

Mechanism Design – Setup – some terminology

Private Values

IPV = independent privat values

- i's preferences depend only on θ_i:

  $u_i(\theta, x) = u_i(\theta_i, x)$
- 'interdependent values' otherwise

Independent Types

- θ_i's distribution indep. of other types θ_{-i}:

  $f(\theta) = \prod_{i=1}^{n} f_i(\theta_i)$
- 'correlated' types otherwise

Quasi-linear Utilities

- outcomes $X = K \times \mathbb{R}^n$, where

  $k \in K$ some physical allocation,

  $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$ transfers
- i's utility is linear in money (his transfer):

  $u_i(\theta, x) = \boxed{v_i(\theta, k)} + t_i$

Social Choice/Unrestricted Domain

- $X = \{a, b, \ldots\}$ finite set of alternatives
- θ_i gives ranking over alternatives:

  $a\theta_i b \Leftrightarrow a \succ_i b$
- Unrestricted domain if

  Θ_i contains all possible rankings over X

Mechanism Design – environment examples

Ex 1. Public good

- outcomes $(k, t) \in X = \{0, 1\} \times \mathbb{R}^n$

  $k \in \{0, 1\}$ with $k = 1$ if bridge is built

  $t_i \in \mathbb{R}$ transfer to agent i
- θ_i is i's willingness to pay for bridge

  $u_i(\theta, x) = \theta_i k + t_i$

private values

quasi-linear utilities

Ex. 2 Allocation with externalities

- outcomes $(k, t) \in X = \{0, 1, \dots, n\} \times \mathbb{R}^n$

  $$k = \begin{cases} 0 & \text{if nobody gets object} \\ i & \text{if agent } i \text{ gets object} \end{cases}$$

  $t_i \in \mathbb{R}$ transfer to agent i
- $\boxed{\theta_i} = (\theta_i^i, \theta_i^x)$ with utility

  $$u_i(\theta, x) = \begin{cases} t_i & \text{if } k = 0 \\ \theta_i^i + t_i & \text{if } k = i \\ -\theta_i^x + t_i & \text{if } k \notin \{0, i\} \end{cases}$$

quasi-linear utility

private values

Mechanism Design – Social Choice Functions

Our goal is generally to choose a good outcome $x \in X$ given the realised preferences $\theta \in \Theta$

Definition (Social Choice Function)

A **social choice function** (scf) $\xi \colon \Theta \to X$ assigns to each type profile $\theta \in \Theta$ an alternative $\xi(\theta) \in X$.

The problem of the mechanism designer is not 'lack of power'

- if the designer knew θ, she could always choose the 'optimal' outcome

The problem is 'just' the asymmetric information

Mechanism Design – Mechanisms (general/indirect)

Typically, social (collective) outcomes are determined through interaction in some institution

Definition (Mechanism)

A **mechanism** $\Gamma = (S_1, \ldots, S_n, g)$ consists of

- a strategy space S_i for each agent i
- an outcome function $g : S_1 \times \cdots \times S_n \to X$.

A mechanism $\Gamma = (S, g)$ together with the environment induces a Bayesian game:

$G_\Gamma = (n,\ \{S_i\}_{i \leq n},\ \{\tilde{u}_i\}_{i \leq n},\ \Theta,\ F)\,,$ with payoffs $\tilde{u}_i(\theta, s_1, \ldots, s_n) = u_i(\theta, g(s_1, \ldots, s_n))$.

Mechanism Design – Incentive Compatibility

We have several solution concepts: Let $(s_i^*)_{i=1}^n$ be a strategy profile, where $\forall i$: $s_i : \Theta_i \to S_i$

- Dominant strategy equilibrium: for all i, θ_i, s_i:

$$u_i((\theta_i, \theta_{-i}), g(s_i^*(\theta_i), s_{-i})) \geq u_i((\theta_i, \theta_{-i}), g(s_i, s_{-i})) \quad \forall \theta_{-i}, s_{-i}$$

- Ex-post equilibrium: for all i, θ_i, s_i:

$$u_i((\theta_i, \theta_{-i}), g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}))) \geq u_i((\theta_i, \theta_{-i}), g(s_i, s_{-i}^*(\theta_{-i}))) \quad \forall \theta_{-i}$$

- Bayes-Nash equilibrium: for all i, θ_i, s_i:

$$\int_{\Theta_{-i}} u_i((\theta_i, \theta_{-i}), g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i})))\ \mathrm{d}F_{-i}(\theta_{-i}|\theta_i)$$
$$\geq \int_{\Theta_{-i}} u_i((\theta_i, \theta_{-i}), g(s_i, s_{-i}^*(\theta_{-i})))\ \mathrm{d}F_{-i}(\theta_{-i}|\theta_i)$$

Mechanism Design – Participation Constraints

Mechanism Design – Implementation

Definition

We say that mechanism $\Gamma = (S, g)$ [...]-**implements** scf ξ
if there exists a [...]-equilibrium strategy profile $(s_i^*)_{i=1}^n$ such that

$$g(s_1^*(\theta_1), \ldots, s_n^*(\theta_n)) = \xi(\theta) \quad \text{for all } \theta \in \Theta.$$

where [...]$\in$ {dominant strategy, ex-post, Bayes}

- Full implementation: every equilibrium results in $\xi(\theta)$
- Partial implementation: there is an equilibrium that results in $\xi(\theta)$

We focus on partial implementation

Mechanism Design – example second-price auction

Mechanism Design – Revelation Principle (direct mechanisms)

Theorem (Revelation Principle)

For any mechanism $\Gamma = (S, g)$ and [...]-equilibrium strategy profile $(s_i^*)_{i=1}^n$ that implements scf ξ, there exists a ***direct*** *mechanism $\hat{\Gamma} = (\Theta, \xi)$ such that* ***truthtelling*** *is a [...]equilibrium.*

- Only ensures that there is AN equilibrium
- In different (indirect) mechanisms sharing the same direct mechanism other equilibria may arise