21-3-22 lecture 03

Last lecture

1. Envelope Thm
2. Revenue equivalence
3. "IC characterisation"

(q,t) is IC

$\iff$

(i) $q(\cdot)$ increasing

(ii) $V(\theta) = V(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} \frac{\partial}{\partial \theta} u(s, q(s))\, ds$

$-t(\theta) = u(\theta, q(\theta)) - \int_{\underline{\theta}}^{\theta} \cdots - V(\underline{\theta})$

Today

- Intro Mech. Design
- Welfare maximising mechanisms

for additional material see

- Krishna p. 303 → & p. 317 →
- Borgess p. 130 → & p. 113 →
- MWG p. 858 → & p. 883 →

The Environment

- n agents
- each agent i has private information (his type) $\theta_i \in \Theta_i$
- set of possible alternatives/outcomes $x \in X$
- each agent is expected-utility maximiser with vNM utility function

$$u_i(\theta, x) \in \mathbb{R}, \qquad \text{for } \theta \in \Theta = \Theta_1 \times \cdots \times \Theta_n \text{ and } x \in X.$$

- the type profile $\theta = (\theta_1, \ldots, \theta_n)$ is distributed according to F with density $f > 0$
- **notation:** we write

$$\boxed{\theta_{-i}} = (\theta_1, \ldots, \theta_{i-1}, \theta_{i+1}, \ldots, \theta_n) \qquad \text{and} \qquad (\theta_i, \theta_{-i}) = \theta$$

θ_i is missing

Mechanism Design – Setup – some terminology

Private Values

- i's preferences depend only on θ_i:

  $$u_i(\theta, x) = u_i(\theta_i, x)$$

- 'interdependent values' otherwise

IPV = independent privat values

Independent Types

- θ_i's distribution indep. of other types θ_{-i}:

  $$f(\theta) = \prod_{i=1}^{n} f_i(\theta_i)$$

- 'correlated' types otherwise

Quasi-linear Utilities

- outcomes $X = K \times \mathbb{R}^n$, where

  $k \in K$ some physical allocation,

  $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$ transfers

- i's utility is linear in money (his transfer):

  $$u_i(\theta, x) = \boxed{v_i(\theta, k)} + t_i$$

Social Choice/Unrestricted Domain

- $X = \{a, b, \ldots\}$ finite set of alternatives
- θ_i gives ranking over alternatives:

  $$a\theta_i b \Leftrightarrow a \succ_i b$$

- Unrestricted domain if

  Θ_i contains all possible rankings over X

Mechanism Design – environment examples

Ex 1. Public good

- outcomes $(k, t) \in X = \{0, 1\} \times \mathbb{R}^n$

  $k \in \{0, 1\}$ with $k = 1$ if bridge is built

  $t_i \in \mathbb{R}$ transfer to agent i
- θ_i is i's willingness to pay for bridge

  $u_i(\theta, x) = \theta_i k + t_i$

private values

quasi-linear utilities

Ex. 2 Allocation with externalities

- outcomes $(k, t) \in X = \{0, 1, \ldots, n\} \times \mathbb{R}^n$

  $$k = \begin{cases} 0 & \text{if nobody gets object} \\ i & \text{if agent } i \text{ gets object} \end{cases}$$

  $t_i \in \mathbb{R}$ transfer to agent i
- $\boxed{\theta_i} = (\theta_i^i, \theta_i^x)$ with utility

  $$u_i(\theta, x) = \begin{cases} t_i & \text{if } k = 0 \\ \theta_i^i + t_i & \text{if } k = i \\ -\theta_i^x + t_i & \text{if } k \notin \{0, i\} \end{cases}$$

quasi-linear utility

private values

Mechanism Design – Social Choice Functions

according to some criteria

Our goal is generally to choose a good outcome $x \in X$ given the realised preferences $\theta \in \Theta$

Definition (Social Choice Function)

A **social choice function** (scf) $\xi \colon \Theta \to X$ assigns to each type profile $\theta \in \Theta$ an alternative $\xi(\theta) \in X$.

The problem of the mechanism designer is not 'lack of power'

- if the designer knew θ, she could always choose the 'optimal' outcome

The problem is 'just' the asymmetric information

Typically, social (collective) outcomes are determined through interaction in some institution

Definition (Mechanism)

A **mechanism** $\Gamma = (S_1, \ldots, S_n, g)$ consists of

- a strategy space S_i for each agent i
- an outcome function $g : S_1 \times \cdots \times S_n \to X$.

A mechanism $\Gamma = (S, g)$ together with the environment induces a Bayesian game:

$G_\Gamma = (n,\ \{S_i\}_{i \leq n},\ \{\tilde{u}_i\}_{i \leq n},\ \Theta,\ F)$, with payoffs $\tilde{u}_i(\theta, s_1, \ldots, s_n) = u_i(\theta, g(s_1, \ldots, s_n))$.

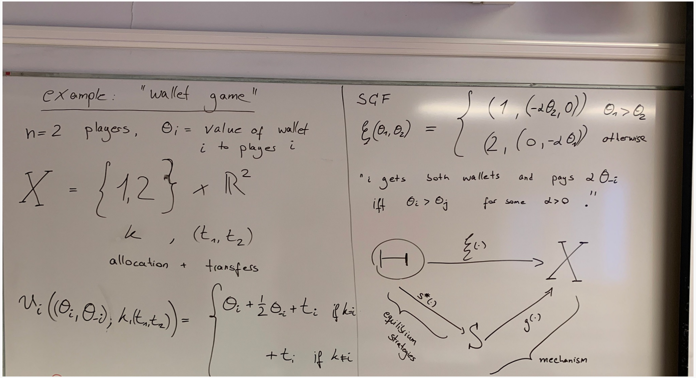
Example for Mechanism: English auction (=ascending-clock auction) with $n=2$ bidders

- A price is publicly displayed
- Price increases continuously from $p_0 = 0$
- Bidder i's strategy: drop out when price reaches $s_i(\theta_i)$
- When i drops out (first), $j \neq i$ wins and pays s_i

with $u_i(\theta,(h,t)) = \begin{cases} \theta_i + \frac{1}{2}\theta_{-i} + t_i & \text{if } h=i \\ +t_i & \text{else} \end{cases}$

see Krishna p. 91 ->

symmetric equilibrium $s_i^*(\theta_i) + (1+\frac{1}{2})\theta_i$


example: "wallet game"
$n=2$ players, θ_i = value of wallet i to player i
$X = \{1,2\} \times \mathbb{R}^2$
k , (t_1, t_2)
allocation + transfers
$v_i((\theta_i, \theta_{-i}), k,(t_1,t_2)) = \begin{cases} \theta_i + \frac{1}{2}\theta_{-i} + t_i & \text{if } k=i \\ + t_i & \text{if } k \neq i \end{cases}$
SCF
$\xi(\theta_1, \theta_2) = \begin{cases} (1, (-\alpha\theta_2, 0)) & \theta_1 > \theta_2 \\ (2, (0, -\alpha\theta_1)) & \text{otherwise} \end{cases}$
"i gets both wallets and pays $\alpha\theta_{-i}$ iff $\theta_i > \theta_j$ for some $\alpha > 0$."
Θ
$\xi(\cdot)$
X
$s^*(\cdot)$
S
$g(\cdot)$
equilibrium strategies
mechanism

Mechanism Design – Incentive Compatibility

We have several solution concepts: Let $(s_i^*)_{i=1}^n$ be a strategy profile, where $\forall i:\ s_i : \Theta_i \to S_i$

- Dominant strategy equilibrium: for all i, θ_i, s_i:

$$u_i((\theta_i, \theta_{-i}), g(s_i^*(\theta_i), s_{-i})) \geq u_i((\theta_i, \theta_{-i}), g(s_i, s_{-i})) \quad \forall \theta_{-i}, s_{-i}$$

strategy s_i^* is optimal no matter...

a) what types my opponents have and

b) what strategies they play.

Mechanism Design – Incentive Compatibility

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- Ex-post equilibrium: for all i, θ_i, s_i:

$$u_i((\theta_i, \theta_{-i}), g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}))) \geq u_i((\theta_i, \theta_{-i}), g(s_i, s_{-i}^*(\theta_{-i}))) \quad \forall \theta_{-i}$$

strategy s_i^* is optimal no matter

(a) what types my opponents have

if they play their equilibrium strategies s_{-i}^*

We have several solution concepts: Let $(s_i^*)_{i=1}^n$ be a strategy profile, where $\forall i$: $s_i : \Theta_i \to S_i$

- Dominant strategy equilibrium: for all i, θ_i, s_i:

$$u_i((\theta_i, \theta_{-i}), g(s_i^*(\theta_i), s_{-i})) \geq u_i((\theta_i, \theta_{-i}), g(s_i, s_{-i})) \quad \forall \theta_{-i}, s_{-i}$$

$\Downarrow$ ($\not\Uparrow$ with private values)

- Ex-post equilibrium: for all i, θ_i, s_i: $\longrightarrow$ english auction equilibrium was ex post

$$u_i((\theta_i, \theta_{-i}), g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}))) \geq u_i((\theta_i, \theta_{-i}), g(s_i, s_{-i}^*(\theta_{-i}))) \quad \forall \theta_{-i}$$

$\Downarrow$

- Bayes-Nash equilibrium: for all i, θ_i, s_i:

$$\int_{\Theta_{-i}} u_i((\theta_i, \theta_{-i}), g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}))) \ \mathrm{d}F_{-i}(\theta_{-i}|\theta_i)$$

$$\geq \int_{\Theta_{-i}} u_i((\theta_i, \theta_{-i}), g(s_i, s_{-i}^*(\theta_{-i}))) \ \mathrm{d}F_{-i}(\theta_{-i}|\theta_i)$$

Mechanism Design – Participation Constraints = "individual rationality" (IR)

Timing of participation decision matters

- Ex-post IR: $u_i((\theta_i, \theta_{-i}); g(s^*(\theta))) \geq 0 \quad \forall\, \theta_i, \theta_{-i}$

  ↳ agent could drop out at any time.

- Ex-interim IR: $\mathbb{E}_{\theta_{-i}}\left[u_i((\theta_i, \theta_{-i}); g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}))) \mid \theta_i \right] \geq 0 \quad \forall\, \theta_i$

  ↳ agent i decides whether to participate after learning own type θ_i but before learning others' types θ_{-i}

- Ex-ante IR: $\mathbb{E}_{(\theta_i, \theta_{-i})}\left[u_i((\theta_i, \theta_{-i}); g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}))) \right] \geq 0$

  ↳ agent decides whether to participate before knowing own type or others' types

Mechanism Design – Implementation

Definition

We say that mechanism $\Gamma = (S, g)$ [...]-**implements** scf ξ
if there exists a [...]-equilibrium strategy profile $(s_i^*)_{i=1}^n$ such that

$$g(s_1^*(\theta_1), \ldots, s_n^*(\theta_n)) = \xi(\theta) \quad \text{for all } \theta \in \Theta.$$

where [...]$\in$ {dominant strategy, ex-post, Bayes}

- Full implementation: every equilibrium results in $\xi(\theta)$
- Partial implementation: there is an equilibrium that results in $\xi(\theta)$

We focus on partial implementation

Mechanism Design – Revelation Principle (direct mechanisms)

dominant, ex-post, Bayesian

Theorem (Revelation Principle)

For any mechanism $\Gamma = (S, g)$ *and [...]-equilibrium strategy profile* $(s_i^*)_{i=1}^n$ *that implements scf* ξ, *there exists a **direct** mechanism* $\hat{\Gamma} = (\Theta, \xi)$ *such that **truthtelling** is a [...]equilibrium.*

- Only ensures that there is AN equilibrium
- In different (indirect) mechanisms sharing the same direct mechanism other equilibria may arise

$$\begin{tikzcd} \Theta \arrow[r, "s^*(\cdot)"] \arrow[rr, bend right, "\xi(\theta) = g(s^*(\theta))"'] & S \arrow[r, "g(\cdot)"] & X \end{tikzcd}$$

Mechanism Design – Revelation Principle – proof

proof of revelation principle for dominant strategy case

- Take some mechanism $\Gamma = (S, g)$ and DIC-equilibrium $(s_i^*)_{i=1}^n$
- By Dominant-strategy-eq. (DIC), we have

$$u_i(\theta_i, \theta_{-i}, g(s_i^*(\theta_i), s_{-i})) \geq u_i(\theta_i, \theta_{-i}; g(\hat{s}_i, s_{-i})) \quad \forall s_{-i}, \theta_{-i} \ \forall \hat{s}_i$$

in particular $u_i(\theta_i, \theta_{-i}, g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}'))) \geq u_i(\theta_i, \theta_{-i}; g(\hat{s}_i, s_{-i}^*(\theta_{-i}'))) \ \forall \theta_{-i}', \theta_{-i} \ \forall \hat{s}_i$

- In direct mechanism $\xi(\theta_i, \theta_{-i}) = g(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}))$

$$\Rightarrow \quad u_i(\theta_i, \theta_{-i}; \xi(\theta_i, \theta_{-i}')) \geq u_i(\theta_i, \theta_{-i}; \xi(\hat{\theta}_i, \theta_{-i}')) \quad \forall \theta_{-i}', \theta_{-i}, \forall \hat{\theta}_i$$

$\Rightarrow$ Direct mechanism with $\xi(\cdot)$ is DIC $\blacksquare$

Mechanism Design – The Gibbard-Satterthwaite Theorem

Recall from micro 3:

Definition (Dictatorial)

An scf $\xi : \Theta_1 \times \cdots \times \Theta_n \to X$ is **dictatorial** if there is an agent $d \in \{1, \ldots, n\}$ such that $\xi(\theta_d, \theta_{-d})$ is always the favourite outcome of type θ_d.

Theorem (Gibbard-Satterthwaite)

Suppose $|X| \geq 3$ and for all i, Θ_i contains all possible preference rankings over X. If scf ξ with $\xi(\Theta) = X$ is strategy proof, then it is dictatorial.

With unrestricted preferences, there is not a lot we can do...

Not hopeless if preferences are more restricted:

- voting/social-choice literature typically focuses on single-peaked preferences
- we will consider quasi-linear utilities and (mostly) private values

Mechanism Design – quasi-linear utility and private value

- Outcomes: $X = K \times \mathbb{R}^n$: $k \in K$ allocation and $(t_1, \ldots, t_n) \in \mathbb{R}^n$ transfers
- Utilities: $u_i(\theta, x) = v_i(\theta_i, k) + t_i$

Note:

- $v_i(\theta_i, k)$ measures the value of allocation k in terms of money
- Utility is transferable across agents through money
- Agents are risk-neutral with respect to money

Mechanism Design – quasi-linear utilities and efficiency

Definition (Pareto efficiency)

An outcome $x = (k, t_1, \dots, t_n) \in X$ is Pareto efficient if there is no other $x' = (k', t'_1, \dots, t'_n) \in X$ such that:

$$\boxed{\sum_{i=1}^{n} t'_i = \sum_{i=1}^{n} t_i} \qquad \text{and} \qquad v_i(\theta_i, k') + t'_i \geq v_i(\theta_i, k) + t_i$$

for all i, with strict inequality for at lease one i.

Proposition

An scf $\xi = (k, t)$ is Pareto efficient if and only if for all $\theta \in \Theta$:

$$\sum_{i=1}^{n} v_i(\theta_i, k(\theta)) \geq \sum_{i=1}^{n} v_i(\theta_i, k') \quad \forall k'.$$

Mechanism Design – VCG Mechanisms

Definition

A Vickrey-Clarke-Groves (VCG) mechanism is given by (k^*, t) where k^* is efficient and

$$t_i(\theta) = \sum_{j \neq i} v_j(\theta_j, k^*(\theta)) + h_i(\theta_{-i}),$$

for some collection of functions $(h_i)_i$ where each h_i is independent of θ_i

Theorem

Truthtelling is a dominant-strategy equilibrium of any VCG mechanism.

Mechanism Design – VCG Mechanism – proof

Take agent i with type θ_i. Truthtelling is optimal

$$v_i(\theta_i, k^*(\theta_i, \theta_{-i})) + t_i(\theta_i, \theta_{-i}) \geq v_i(\theta_i, k^*(\hat{\theta}_i, \theta_{-i})) + t_i(\hat{\theta}_i, \theta_{-i}) \quad \forall \hat{\theta}_i \neq \theta_i$$

insert for t_i from definition of VCG

$$t_i(\theta) = \sum_{j \neq i} v_j(\theta_j, k^*(\theta)) + h_i(\theta_{-i}),$$

$$\Leftrightarrow v_i(\theta_i, k^*(\theta_i, \theta_{-i})) + \sum_{j \neq i} v_j(\theta_j, k^*(\theta_i, \theta_{-i})) + h_i(\theta_{-i})$$

$$\geq v_i(\theta_i, k^*(\hat{\theta}_i, \theta_{-i})) + \sum_{j \neq i} v_j(\theta_j, k^*(\hat{\theta}_i, \theta_{-i})) + h(\theta_{-i})$$

$$= \underbrace{\sum_{j=1}^{n} v_j(\theta_j, k^*(\theta_i, \theta_{-i})) \geq \sum_{j=1}^{n} v_j(\theta_j, k^*(\hat{\theta}_i, \theta_{-i}))}_{\text{holds because } k^* \text{ is efficient}} \quad \square$$