22-03-22 Lecture 04

yesterday

- A lot of definitions:
  - environments
  - mechanisms
  - SCF
  - direct mechanisms
  - incentive compatibility (IC) (solution concept)
  - individual rationality (IR) (participation constraint)
  - VCG mechanisms
- And two results
  - Revelation Principle
  - VCG mechanisms implement efficient allocation DIC

Today

- A special VCG mechanism
  → "Pivot mechanism
- Bayesian IC & efficiency
  → expected externality mechanism
- Myerson-Satterthwaite-Theorem

Mechanism Design – VCG Mechanisms

A special case of VCG mechanisms is the pivot mechanism (or Clarke mechanism):

Definition (pivot mechanism)

A pivot mechanism is a VCG mechanism with

$$h_i(\theta_{-i}) = -\sum_{j \neq i} v_j(\theta_j, k^*_{-i}(\theta_{-i})),$$

where $k^*_{-i}(\theta_{-i})$ is an efficient alternative for the $n-1$ agents different from i

- Each agent pays the externality imposed on other agents:

$$t_i(\theta) = \underbrace{\sum_{j \neq i} v_j(\theta_j, k^*(\theta))}_{\text{if } i \text{ is present}} - \underbrace{\sum_{j \neq i} v_j(\theta_j, k^*_{-i}(\theta_{-i}))}_{\text{if } i \text{ is not present}}.$$

- If adding agent i with type θ_i does not change allocation, then $t_i = 0$
- The second-price auction is a pivot mechanism ← showing that truthtelling DIC ↳ exercise

Mechanism Design – VCG Mechanisms

- Is there an ex-post efficient mechanism that is DIC but not a VCG mechanism?
- If the environment is 'rich' enough, the answer is no:

  Let $\mathcal{V}$ denote the set of all possible functions from K to $\mathbb{R}$

Theorem

If for all agents i, the set of preferences is such that $\{v_i(\theta_i, \cdot)\}_{\theta_i \in \Theta_i} = \mathcal{V}$, then every direct mechanism in which truthtelling is a dominant strategy is a VCG-mechanism.

Mechanism Design – DIC and efficiency

Ex post efficiency and DIC is 'almost equivalent' to VCG mechanism

That is great because...

- these are simple to characterise
- we can simply check for the best VCG mechanism in each situation

However,...

- they potentially require large transfers
- we have ignored participation constraints
- they are generally not budget balanced: $\sum_i t_i(\theta) \neq 0$

Mechanism Design – Bayesian incentive compatibility

What if we weaken our solution concept and look at Bayesian Mechanism Design?

- We will focus in the independent case: $f(\theta) = \prod_i f_i(\theta_i)$

Recall: truthtelling is a Bayes-Nash equilibrium if for all i and all θ_i:

$$\mathbb{E}_{\theta_{-i}}\left[v_i(\theta_i, k(\theta_i, \theta_{-i})) + t_i(\theta_i, \theta_{-i})\right] \geq \mathbb{E}_{\theta_{-i}}\left[v_i(\theta_i, k(\hat{\theta}_i, \theta_{-i})) + t_i(\hat{\theta}_i, \theta_{-i})\right] \quad \forall \hat{\theta}_i \qquad \text{(BIC)}$$

We hope that we can exploit weakened IC requirement (now only in expectation over θ_{-i}) to eliminate some undesirable features of VCG mechanisms.

- ...and indeed we can
- ...at first sight

Mechanism Design – Expected Externality Mechanism

- Let k^* be an ex-post efficient allocation rule
- Consider the following transfers:

see MWG p. 885 & exercise

$$t_i(\theta_i, \theta_{-i}) = \mathbb{E}_{\tilde{\theta}_{-i}}\left[\sum_{j \neq i} v_j(\tilde{\theta}_j, k^*(\theta_i, \tilde{\theta}_{-i}))\right] + h_i(\theta_{-i}),$$

with

$$h_i(\theta_{-i}) = -\frac{1}{n-1}\sum_{j \neq i}\mathbb{E}_{\tilde{\theta}_{-j}}\left[\sum_{\ell \neq j} v_\ell(\tilde{\theta}_\ell, k^*(\theta_j, \tilde{\theta}_{-j}))\middle|\,\theta_j\right].$$

Definition (Expected Externality Mechanism)

The mechanism (k^*, t) defined above is called Expected Externality Mechanism.

Mechanism Design – Expected Externality Mechanism

Proposition

The Expected Externality Mechanism is budget balanced and truthtelling is BIC.

That is

$$\begin{aligned}
&\sum_{i=1}^{n} t_i(\theta) \\
&= \sum_{i=1}^{n} \mathbb{E}_{\tilde{\theta}_{-i}} \left[\sum_{j \neq i} v_j(\tilde{\theta}_j, k^*(\theta_i, \tilde{\theta}_{-i})) \right] - \frac{1}{n-1} \sum_{i=1}^{n} \sum_{j \neq i} \mathbb{E}_{\tilde{\theta}_{-j}} \left[\sum_{\ell \neq j} v_\ell(\tilde{\theta}_\ell, k^*(\theta_j, \tilde{\theta}_{-j})) \right] \\
&= 0.
\end{aligned}$$

Mechanism Design – Expected Externality Mechanism

- Expected Externality mechanism achieves budget balance
- but did we really gain that much?

  the following result suggests no:

additional material
Mailath p. 321 →
Krishna p. 75 →

Theorem

Fix an ex-post efficient allocation rule k^* and a BIC mechanism that implements k^*. Then there exist constants h_i such that the VCG mechanism with transfer rule

$$t_i(\theta) = \sum_{j \neq i} v_j(\theta_j, k^*(\theta)) + h_i$$

gives each player the same interim payoff. – as in the BIC mechanism.

↳ expected payoff when knowing own type but before knowing others' types.

Mechanism Design – BIC and Efficiency

To sum up:

- VCG mechanisms give us a pretty complete picture of the expected utilities that can be achieved in incentive compatible and efficient mechanisms
- With expected externality mechanism we can achieve budget balance ex post

But...

- We still completely ignored participation constraints
- ...and that is generally problematic as we see now

Mechanism Design – Bilateral Trade

Question: Is efficient trade possible when both sides have private information?

- single indivisible good
- one buyer with $\theta \in [\underline{\theta}, \bar{\theta}]$ drawn from F
- one seller with production cost $c \in [\underline{c}, \bar{c}]$ drawn from G
- trade is efficient sometimes: $\underline{c} < \bar{\theta}$ but not always: $\underline{\theta} < \bar{c}$

Theorem (Myerson-Satterthwaite)

There is no ex-post efficient, budget balanced, BIC mechanism that satisfies interim IR for buyer and seller.

Direct Mechanism:

- $q(\theta, c) \in [0, 1]$ = prob. of trade → efficent $q^*(\theta,c) = \mathbb{1}\{\theta > c\}$
- $t_B(\theta, c)$ transfer to buyer $t_S(\theta, c)$ transfer to seller

The buyer's expected utility from report $\hat{\theta}$ is

$$\int_{\underline{c}}^{\bar{c}} \left(\theta q(\hat{\theta}, c) + t_B(\hat{\theta}, c)\right) \mathrm{d}G(c)$$

Define: interim expected probability of trade; interim expected transfer

- $Q_B(\hat{\theta}) = \int_{\underline{c}}^{\bar{c}} q(\hat{\theta}, c)\, \mathrm{d}G(c)$ and $T_B(\hat{\theta}) = \int_{\underline{c}}^{\bar{c}} t_B(\hat{\theta}, c)\, \mathrm{d}G(c)$ for buyer
- $Q_S(\hat{c}) = \int_{\underline{\theta}}^{\bar{\theta}} q(\theta, \hat{c})\, \mathrm{d}F(\theta)$ and $T_S(\hat{c}) = \int_{\underline{\theta}}^{\bar{\theta}} t_S(\theta, \hat{c})\, \mathrm{d}F(\theta)$ for seller

Incentive compatibility (Bayesian):

$$\theta Q_B(\theta) + T_B(\theta) \geq \theta Q_B(\hat{\theta}) + T_B(\hat{\theta}) \qquad (BIC_{buyer})$$

$$T_S(c) - cQ_S(c) \geq T_S(\hat{c}) - cQ_S(\hat{c}) \qquad (BIC_{seller})$$

Individual rationality (interim):

$$\theta Q_B(\theta) + T_B(\theta) \geq 0 \qquad (IR_{buyer})$$

$$T_S(c) - cQ_S(c) \geq 0 \qquad (IR_{seller})$$

Budget Balance holds if $t_B(\theta, c) + t_S(\theta, c) \leq 0$, we will require a weaker condition:

$$\int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{c}}^{\bar{c}} (t_B(\theta, c) + t_S(\theta, c)) \, \mathrm{d}F(\theta) \, \mathrm{d}G(c) \leq 0 \qquad (BB)$$

no money from outside in expectation

we will show

no mechanism with ex-post efficient trade ($q(\theta, c) = \mathbb{1}_{\{\theta > c\}}$) satisfies these conditions

Mechanism Design – Bilateral Trade – proof of Myerson Satterthwaite

We can apply screening results to expected terms Q and T to conclude

Lemma

Suppose (q, t_B, t_S) satisfies BIC_{buyer} and BIC_{seller}, then

1. $Q_B(\theta)$ *is non-decreasing*
2. $Q_S(c)$ *is non-increasing*
3. $V_B(\theta) = V_B(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} Q_B(s)\,\mathrm{d}s$
4. $V_S(c) = V_s(\bar{c}) + \int_{c}^{\bar{c}} Q_S(s)\,\mathrm{d}s$

Mechanism Design – Bilateral Trade – proof of Myerson Satterthwaite

Since we are interested in ex-post efficient allocations: recall the following theorem:

Theorem

Fix an ex-post efficient allocation rule k^* and a BIC mechanism that implements k^*. Then there exist constants h_i such that the VCG mechanism with transfer rule

$$t_i(\theta) = \sum_{j \neq i} v_j(\theta_j, k^*(\theta)) + h_i$$

gives each player the same interim payoff. ⟶ here: same $V_B(\theta)$ same $V_S(c)$ for all θ, c

With constants h_B and h_S, VCG implies the following transfer rules:

$$t_B(\theta, c) = \begin{cases} -c + h_B & \text{if } \theta > c \\ h_B & \text{otherwise} \end{cases} \quad \text{and} \quad t_S(\theta, c) = \begin{cases} \theta + h_S & \text{if } \theta > c \\ h_S & \text{otherwise.} \end{cases}$$

The (interim) expected utility of the buyer is then

$$V_B(\theta) = \int_{\underline{c}}^{\bar{c}} \left((\theta - c)\, \mathbb{1}_{\{\theta > c\}} + h_B \right) \mathrm{d}G(c) = \int_{\underline{c}}^{\bar{c}} (\theta - c)\, \mathbb{1}_{\{\theta > c\}}\, \mathrm{d}G(c) + h_B$$

Mechanism Design – Bilateral Trade – proof of Myerson Satterthwaite

Considering the ex-ante expected utility of the buyer

$$\int_{\underline{\theta}}^{\bar{\theta}} V_B(\theta)\,\mathrm{d}F(\theta) = \underbrace{\int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{c}}^{\bar{c}} (\theta - c)\,\mathbb{1}_{\{\theta > c\}}\,\mathrm{d}G(c)\,\mathrm{d}F(\theta)}_{=\text{ex-ante surplus from efficient trade } \equiv \mathcal{S}} + h_B = \mathcal{S} + h_B.$$

Same steps for the seller

$$\int_{\underline{c}}^{\bar{c}} V_S(c)\,\mathrm{d}G(c) = \mathcal{S} + h_S.$$

However, by Budget Balance (we don't inject money from outside) it must be that

$$\int_{\underline{\theta}}^{\bar{\theta}} V_B(\theta)\,\mathrm{d}F(\theta) + \int_{\underline{c}}^{\bar{c}} V_S(c)\,\mathrm{d}G(c) \leq \mathcal{S}.$$

Mechanism Design – Bilateral Trade – proof of Myerson Satterthwaite

Assume that $\underline{\theta} = \underline{c}$ and $\bar{\theta} = \bar{c}$ → not needed for result but simplifies proof steps

- Efficient allocation ($q^*(\theta, c) = \mathbb{1}_{\{\theta > c\}}$) implies that

  (i) $Q_B(\underline{\theta}) = \int_{\underline{c}}^{\bar{c}} \mathbb{1}_{\{\underline{\theta} > c\}} \, dG(c) = 0$

  (ii) $Q(\bar{c}) = 0$

- Then IR_{Buyer} at $\underline{\theta}$: $\underline{\theta} \cdot Q_B(\underline{\theta}) + T_B(\underline{\theta}) = 0 + h_B \geq 0.$

  IR_{seller} at $\bar{c}$: $T_S(\bar{c}) = h_S \geq 0.$

→ This gives a contradiction!

$$\int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{c}}^{\bar{c}} (V_B(\theta) + V_S(c)) \, dG(c) \, dF(\theta) = 2S + h_B + h_S$$

but also $\leq S$ by Budget Balance ↯ □

Mechanism Design – Bilateral Trade – Recap

- Ex-post efficient trade is not feasible
- Note: what we showed implies that trade is ex-post inefficient in every equilibrium of any bargaining game with voluntary particiation