28-3-22 lecture 5

last week

- Efficient (= welfare max.) mechanisms
  - VCG mechanisms for DIC (pivot mechanism in particular)
  - Expected externality mechanism for BIC
- Budget Balance and IR ?

  Myerson–Satterthwaite Thm

today

Revenue maximisation

"optimal auctions"

Mechanism Design – Revenue Maximisation

The auction problem:

- single indivisible object
- seller cost c
- n potential buyers with type θ_i
- utility $\theta_i q_i + t_i$
- types are independently distributed on $[\underline{\theta}_i, \bar{\theta}_i]$ according to F_i with density $f_i > 0$
- feasible allocation probabilities: $q_i(\theta) \in [0,1]$ with $\sum_{i=1}^n q_i(\theta) \leq 1$

Seller commits to mechanism $(q,t) \colon \Theta \to [0,1]^n \times \mathbb{R}^n$ to maximise revenue

$$\mathbb{E}_\theta\left[- \sum_{i=1}^{n} t_i(\theta) - c \cdot \sum_{i=1}^{n} q_i(\theta) \right]$$

Mechanism Design – Revenue Maximisation – Optimal Auctions

Let's compare different auction formats for the example $n = 2$, $\theta_i \overset{iid}{\sim} U([0,1])$, $c = 0$

1. **First-price auction**
   - Each bidder makes a bid $b = \boxed{s^{\text{FPA}}(\theta_i)}$. The highest bid wins. The winner pays his bid.
   - Find the symmetric equilibrium bid function s^{FPA} (hint: linear function)
2. **All-pay auction (=contest)**
   - Each bidder makes a bid $b = s^{\text{APA}}(\theta_i)$. The highest bid wins. Each bidder pays his bid.
   - Find the symmetric equilibrium bid function s^{APA}.
3. **English auction (=ascending-clock auction)**
   - A price is publicly displayed and increases continuously from $p_0 = 0$.
     Bidder i drops out when the price reaches $s^{\text{SPA}}(\theta_i)$.
     When i drops out (first), $j \neq i$ wins and pays p at which i dropped out.
   - What is the weakly dominant stopping strategy s^{SPA}?

Q: **What is the expected revenue of the seller?**

First-price - auction

• What is bidder 1's expected payoff if his valuation is θ_1, bidder 2 uses bidding strategy $s(\cdot)$, and bidder 1 bids b_1?

$$(\theta_1 - b_1)\cdot \mathbb{P}_r\left[s(\theta_2) \le b_1\right]$$

$$= (\theta_1 - b_1)\,\underbrace{\mathbb{P}_r\left[s^{-1}(s(\theta_2)) \le s^{-1}(b_1)\right]}_{\mathbb{P}_r\left[\theta_2 \le s^{-1}(b_1)\right] = s^{-1}(b_1)}$$

$$\max_{b_1}\left\{(\theta_1 - b_1)\cdot s^{-1}(b_1)\right\}$$

$$\text{FOC: } \frac{\partial}{\partial b_1}\{\ldots\} = 0 \iff -s^{-1}(b_1) + \frac{\theta_1 - b_1}{s'(s^{-1}(b_1))} = 0$$

$$\iff -\theta_1 + \frac{\theta_1 - s(\theta_1)}{s'(\theta_1)} = 0$$

$$\iff s'(\theta_1)\cdot\theta_1 = \theta_1 - s(\theta_1) \longrightarrow s^{FPA}(\theta) = \tfrac{1}{2}\theta$$

Inverse function Thm

$$\frac{\partial}{\partial b} f^{-1}(b) = \frac{1}{f'(f^{-1}(b))}$$

$s(\cdot)$ linear

$s(\theta) = a + m\cdot\theta$

$s(0)$ must be 0

$\Rightarrow a = 0$

$\Rightarrow s(\theta) = m\theta$

$\hookrightarrow s^{FPA}(\theta) = \tfrac{1}{2}\theta$

expected revenue

FPA:

$$\mathbb{E}_{\theta_1,\theta_2}\left[\max\left\{\tfrac{1}{2}\theta_1, \tfrac{1}{2}\theta_2\right\}\right]$$

$$= \frac{1}{2}\int_0^1 \left(1 - F^2(\theta)\right)d\theta = \frac{1}{3}$$

All-pay - auction

• What is bidder 1's expected payoff if his valuation is θ_1, bidder 2 uses bidding strategy $s(\cdot)$, and bidder 1 bids b_1?

$$\theta_1 \Pr[s(\theta_2) \le b_1] - b_1$$

$$= \left\{\theta_1 \cdot \underbrace{\Pr[\theta_2 \le s^{-1}(b_1)]}_{= s^{-1}(b_1)} - b_1\right\}$$

$$\frac{\partial}{\partial b_1}\{..\} = \frac{\theta_1}{s'(\theta_1)} - 1 = 0 \iff s'(\theta_1) = \theta_1$$

$$\longrightarrow s^{APA}(\theta) = \frac{1}{2}\theta^2$$

expected revenue

FPA:

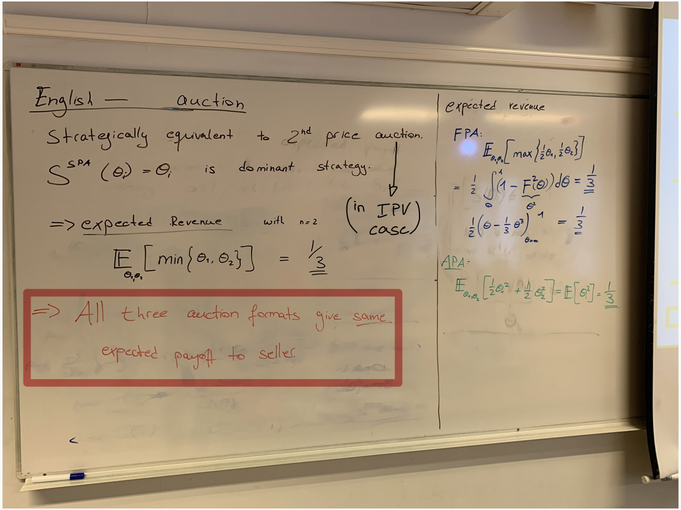
$$\mathbb{E}_{\theta_1,\theta_2}\left[\max\left\{\tfrac{1}{2}\theta_1, \tfrac{1}{2}\theta_2\right\}\right]$$

$$= \frac{1}{2}\int_0^1 \left(1 - \underbrace{F^2(\theta)}_{\theta^2}\right) d\theta = \frac{1}{3}$$

$$\frac{1}{2}\left(\theta - \frac{1}{3}\theta^3\right)\Big|_{\theta=0}^{1} = \frac{1}{3}$$

APA:

$$\mathbb{E}_{\theta_1,\theta_2}\left[\tfrac{1}{2}\theta_1^2 + \tfrac{1}{2}\theta_2^2\right] = \mathbb{E}[\theta_1^2] = \frac{1}{3}$$


English — auction
Strategically equivalent to 2nd price auction.
$S^{SPA}(\theta_i) = \theta_i$ is dominant strategy.
(in IPV case)
=> expected Revenue with n=2
$\mathbb{E}_{\theta_1,\theta_2}[\min\{\theta_1, \theta_2\}] = \frac{1}{3}$
=> All three auction formats give same expected payoff to seller.
expected revenue
FPA:
$\mathbb{E}_{\theta_1,\theta_2}[\max\{\frac{1}{2}\theta_1, \frac{1}{2}\theta_2\}]$
$= \frac{1}{2}\int_0^1 (1 - \underbrace{F^2(\theta)}_{\theta^2})\, d\theta = \frac{1}{3}$
$\frac{1}{2}(\theta - \frac{1}{3}\theta^3)\Big|_{\theta=0}^{1} = \frac{1}{3}$
APA:
$\mathbb{E}_{\theta_1,\theta_2}[\frac{1}{2}\theta_1^2 + \frac{1}{2}\theta_2^2] = \mathbb{E}[\theta^2] = \frac{1}{3}$