Exercise: Regulating a monopolist

A monopolist operates in an industry with inverse demand function $p(q) = 1 - 2q$.
The monopolist has cost $c(\theta, q) = \frac{2}{3}\theta q$, where her private type θ is uniformly distributed on $[0, 1]$. In addition, producing q units of the good on a technology of type θ causes environmental harm $\frac{1}{3}\theta q$.
A regulator is trying to maximise the expectation of consumer surplus plus taxes net of environmental harm:

$$q^2 + \tau - \frac{1}{3}\theta q,$$

by offering a menu $(q(\cdot), \tau(\cdot))$ of the quantity the monopolist must produce and taxes paid by the monopolist. The monopolist can shut down to get payoff of 0 after learning θ.

a) State the regulator's problem that characterises the optimal direct mechanism.

b) Show that if $(q(\cdot), \tau(\cdot))$ is incentive compatible, then $q(\cdot)$ must be decreasing and $V(\theta) = V(1) + \int_\theta^1 \frac{2}{3}q(s)\,\mathrm{d}s$.

c) Solve for the optimal mechanism.

a) Regulator's problem: here $f(\theta) = 1$

$$\max_{\substack{q:[0,1]\to\mathbb{R}_+ \\ \tau:[0,1]\to\mathbb{R}}} \left\{ \int_0^1 \left(q^2(\theta) + \tau(\theta) - \frac{1}{3}\theta\, q(\theta)\right) f(\theta)\, d\theta \right\}$$

s.t.

$$\underbrace{-\frac{2}{3}\theta\, q(\theta) + \overbrace{(1-2q(\theta))}^{p(q(\theta))}\, q(\theta) - \tau(\theta)}_{} \geq -\frac{2}{3}\theta\, q(\theta') + (1-2q(\theta'))\,q(\theta') - \tau(\theta')$$

$\forall \theta, \theta'$: $IC_{\theta,\theta'}$

$\forall \theta$: IR_θ ___ " ______ ≥ 0

b) (i) $q(\cdot)$ decreasing: take $(IC_{\theta,\theta'})$ & $(IC_{\theta',\theta})$

$$(IC_{\theta,\theta'}) : -\frac{2}{3}\theta\, q(\theta) + \underline{(1-2q(\theta))\,q(\theta)} \,\underline{-\tau(\theta)} \geq -\frac{2}{3}\theta\, q(\theta') + \underline{(1-2q(\theta'))\,q(\theta')} - \underline{\tau(\theta')}$$

$$(IC_{\theta',\theta}) : -\frac{2}{3}\theta'\, q(\theta') + \underline{(1-2q(\theta'))\,q(\theta')} - \underline{\tau(\theta')} \geq -\frac{2}{3}\theta'\, q(\theta) + \underline{(1-2q(\theta))\,q(\theta)} - \underline{\tau(\theta)}$$

b(i) cont'd

Add both sides of $(IC_{\theta,\theta'})$ & $(IC_{\theta',\theta})$ & eliminate common terms

$$\Leftrightarrow -\frac{2}{3}\theta q(\theta) - \frac{2}{3}\theta' q(\theta') \geq -\frac{2}{3}\theta q(\theta') - \frac{2}{3}\theta' q(\theta)$$

$$\Leftrightarrow \theta\left(q(\theta') - q(\theta)\right) \geq \theta'\left(q(\theta') - q(\theta)\right)$$

$$\Leftrightarrow 0 \geq (\theta' - \theta)\left(q(\theta') - q(\theta)\right) \rightsquigarrow q(\cdot) \text{ (weakly) decreasing}$$

b)(i) $q(\cdot)$ decreasing: take $(IC_{\theta,\theta'})$ & $(IC_{\theta',\theta})$

$$(IC_{\theta,\theta'}): -\frac{2}{3}\theta q(\theta) + (1-2q(\theta))q(\theta) - \tau(\theta) \geq -\frac{2}{3}\theta q(\theta') + (1-2q(\theta'))q(\theta') - \tau(\theta')$$

$$(IC_{\theta',\theta}): -\frac{2}{3}\theta' q(\theta') + (1-2q(\theta'))q(\theta') - \tau(\theta') \geq -\frac{2}{3}\theta' q(\theta) + (1-2q(\theta))q(\theta) - \tau(\theta)$$

b (ii) $V(\theta) = V(1) + \int_\theta^1 \frac{2}{3} q(s)\,ds$

$$V(\theta) = \max_{\theta'} \underbrace{\left\{ -\frac{2}{3}\theta q(\theta') + (1-2q(\theta'))q(\theta') - \tau(\theta') \right\}}_{g(\theta,\theta')}$$

$g(\theta,\theta')$ is differentiable in θ and piece-wise linear in θ (for fixed $q(\theta')$ & $\tau(\theta')$)

$V(\theta)$ is pointwise max. of linear functions & therefore convex.

* Any convex function on $(0,1)$ is absolutely contin.

V must be differentiable almost everywhere

with $V(\theta) = -\int_\theta^1 V'(s)\,ds + V(1)$

b (ii) $V(\theta) = V(1) + \int_\theta^1 \frac{2}{3} q(s)\,ds$ cont'd: $V'(\theta) = ?$

Take $IC_{\theta,\theta'}$: $\underbrace{-\frac{2}{3}\theta q(\theta) + (1-2q(\theta))q(\theta) - \tau(\theta)}_{V(\theta)}$

$$\geq -\frac{2}{3}\theta q(\theta') + \underbrace{(1-2q(\theta'))q(\theta') - \tau(\theta') - \frac{2}{3}\theta' q(\theta')}_{=V(\theta')} + \frac{2}{3}\theta' q(\theta')$$

$$\Leftrightarrow V(\theta) - V(\theta') \geq -\frac{2}{3} q(\theta')(\theta - \theta')$$

$$\lim_{\theta' \nearrow \theta} q(\theta') \overset{?}{=} q(\theta)$$

(i) if $\theta > \theta'$: $\frac{V(\theta) - V(\theta')}{\theta - \theta'} \geq -\frac{2}{3} q(\theta')$

(ii) if $\theta < \theta'$: $\frac{V(\theta) - V(\theta')}{\theta - \theta'} \leq -\frac{2}{3} q(\theta')$

$\theta \searrow \theta'$ $\longrightarrow$ $V'(\theta'+) \geq -\frac{2}{3} q(\theta')$

$\theta \nearrow \theta'$ $\longrightarrow$ $V'(\theta'-) \leq -\frac{2}{3} q(\theta')$

If V is differentiable at θ', then $V'(\theta') = V'(\theta'+) = V'(\theta'-) = -\frac{2}{3} q(\theta')$

It follows from def. of $V(\theta)$ that

$$\tau(\theta) = -\frac{2}{3}\theta q(\theta) + (1-2q(\theta))q(\theta) - V(1) - \int_\theta^1 \frac{2}{3} q(s)\,ds = -V(\theta)$$

c) optimal mech. insert $\tau(\theta)$ $(*)$ into objective

$$\max_{q:[0,1]\to\mathbb{R}_+} \left\{ \int_0^1 \Big(q(\theta)^2 \underbrace{- \tfrac{2}{3}\theta q(\theta) + (1-2q(\theta))q(\theta) - V(1) - \int_\theta^1 \tfrac{2}{3} q(s)ds}_{=\tau(\theta)} - \tfrac{\theta}{3} q(\theta) \Big) f(\theta)\, d\theta \right\}$$

s.t. q weakly decreasing (IC) & $V(1) \geq 0$ (IR)

(i) $V(1) = 0$ optimally.

$0 \leq \theta \leq s \leq 1$

(ii) To allow for pointwise max. want to transform $\int_0^1 \Big(\int_\theta^1 \frac{2}{3} q(s) ds \Big) f(\theta) d\theta = (\iint)$

change order of integration $(\iint) = \int_0^1 \underbrace{\int_0^s f(\theta) d\theta}_{=F(s)} \frac{2}{3} q(s) ds = \int_0^1 F(s) \cdot \frac{2}{3} q(s) ds \overset{\text{"}s \rightsquigarrow \theta\text{"}}{=} \int \frac{F(\theta)}{f(\theta)} \frac{2}{3} q(\theta) f(\theta) d\theta$

objective: $\{ \} = \int_0^1 \Big(q(\theta)^2 - \frac{2}{3}\theta q(\theta) + (1-2q(\theta))q(\theta) - \frac{2}{3} q(\theta) \frac{F(\theta)}{f(\theta)} - \frac{1}{3}\theta q(\theta) \Big) f(\theta) d\theta$

$F(\theta) = \theta$
$f(\theta) = 1$

$$= \int_0^1 \Big(-q(\theta)^2 - \frac{5}{3}\theta q(\theta) + q(\theta) \Big) d\theta$$

c) optimal mech.

$$\frac{\partial \{\}}{\partial q(\theta)} \stackrel{!}{=} 0 \quad \Rightarrow \quad q^*(\theta) = \begin{cases} \frac{1-\frac{5}{3}\theta}{2} & \text{if } \theta \leq 3/5 \\ 0 & \text{if } \theta > 3/5 \end{cases}$$

$$\tau^*(\theta) = \begin{cases} -\frac{1}{10} + \frac{5}{6}\theta - \frac{10}{9}\theta^2 & \text{if } \theta \leq \frac{3}{5} \\ 0 & \text{if } \theta > \frac{3}{5}. \end{cases}$$

(τ^* from $\tau(\theta) = -\frac{2}{3}\theta q(\theta) + (1-2q(\theta))q\theta - V(1) - \frac{2}{3}\int_\theta^1 q(s)\,ds$

at $q = q^*$ and $V(1) = 0$.)

$$= \int_0^1 \left(-q(\theta)^2 - \frac{5}{3}\theta q(\theta) + q(\theta)\right) d\theta$$