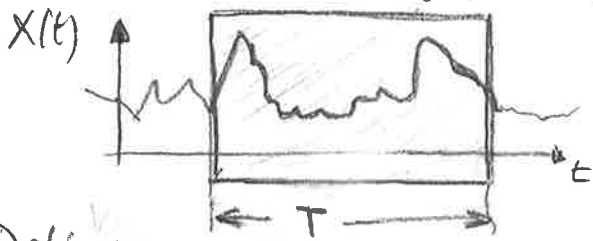


# Elements of noise theory for electrical circuits

- Noise: determines the smallest signal that you can measure, so it limits the dynamical range from below



$X(t)$  = random variable, time-dependent

- We will work with ergodic, stationary processes.

Define:

• average of  $X$ :  $\bar{X} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t) dt$  T = time window

• average power of  $X$ :  $\overline{X^2} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} X^2(t) dt$

For example, you have a resistor; then

$$\frac{\overline{V^2}}{R} = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} \frac{V^2(t)}{R} dt$$

↓  
instantaneous power

• autocorrelation function:

$$R_{XX}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} X(t) X(t+\tau) dt$$

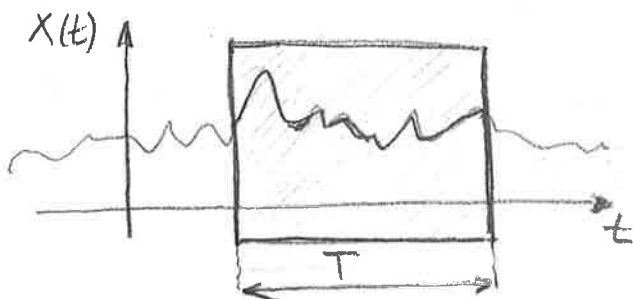
- obviously:

•  $\overline{X^2} = R_{XX}(\tau=0) \geq |R_{XX}(\tau)|$  for  $|\tau| > 0$

•  $R_{XX}(\tau) = R_{XX}(-\tau)$  - this is an even function!

• power spectral density (PSD):

$$S_X(f)$$



→ collect data in a window T from your oscilloscope -

- then take the Fourier transform  $X(f)$

$$S_X(f) \equiv \lim_{T \rightarrow \infty} \frac{|X(f)|^2}{T}$$

$\frac{|X(f)|^2}{T}$  = periodogram (provides an estimate of  $S_X(f)$ )

Units: if  $X$  = Voltage, then  $V^2 Hz^{-1}$ .

• What is the relation between  $S_x(f)$  and the autocorrelation function?

## The Wiener-Khinchin theorem

$$S_x(f) = \mathcal{F}[R_{xx}(\tau)] = \int_{-\infty}^{\infty} R_{xx}(\tau) e^{-2\pi i f \tau} d\tau$$

So the PSD = the Fourier transform of the autocorrelation!

or:

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} df e^{2\pi i f \tau} S_x(f)$$

- note how interesting this is! if you measure  $S_x(f)$  you have access to any correlation (at any  $\tau$ ).

For  $\tau = 0$ ,

$$\overline{x^2} \equiv R_{xx}(0) = \int_{-\infty}^{\infty} df \cdot S_x(f)$$

We will not prove the Wiener-Khinchin theorem (it's not difficult -- if you are interested you can find the proof in textbooks)

\* but let us give an alternative justification based on Parseval's theorem:

$$\int_{-\infty}^{\infty} dt x^2(t) = \int_{-\infty}^{\infty} df |X(f)|^2$$

$\downarrow$  power calculated in time-domain  $\equiv$  power calculated in frequency-domain  $\downarrow$

← Prove it!  
It's a good exercise

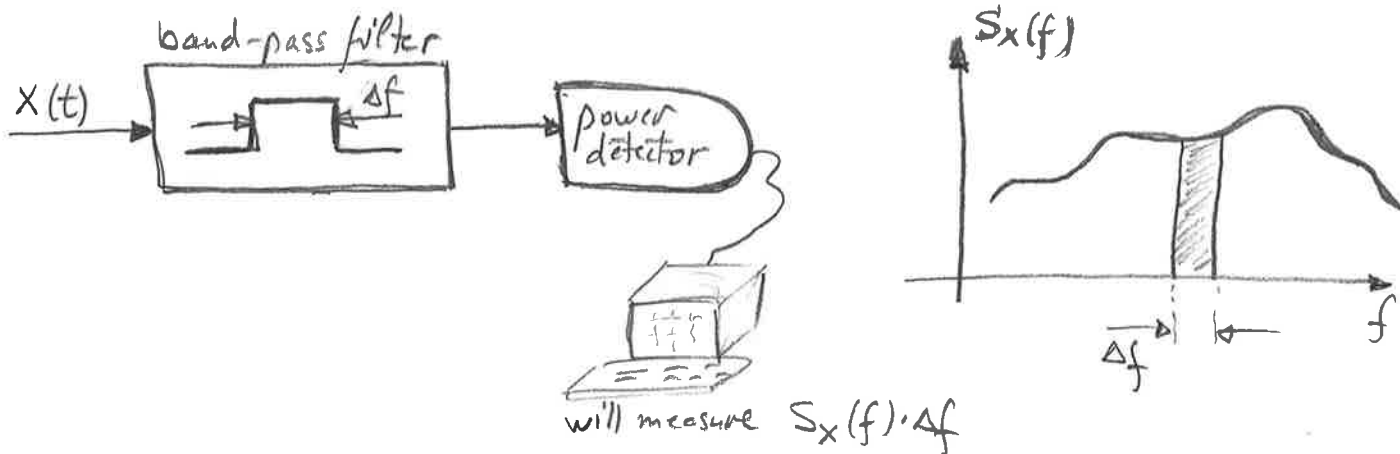
$$\begin{aligned} \text{So } \overline{x^2} &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt x^2(t) \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^{\infty} df |X(f)|^2 \\ &= \int_{-\infty}^{\infty} df \left( \lim_{T \rightarrow \infty} \frac{|X(f)|^2}{T} \right) = \int_{-\infty}^{\infty} df \cdot S_x(f) \end{aligned}$$

So

$$\overline{x^2} = \int_{-\infty}^{\infty} df S_x(f)$$

• What does this mean physically?

$S_x(f) \cdot df = \text{power in the frequency interval } df$



However, there is one little inconvenient.

We can always have time taken at least approximately from  $-\infty$  to  $+\infty$ . But what does it mean to have a negative frequency?

All generators that we have in the lab produce a "positive" frequency.

To bypass this issue it is convenient to restrict ourselves only to positive frequencies, which is what we will do in the rest of the lecture.

How do we do this? Formally, note that for classical fields  $X(t)$

$$X(-\omega) = X^*(\omega) \text{ because } X(t) \text{ is real,} \\ \text{therefore } S(f) = S(-f).$$

$$\text{So } \overline{X^2} = \int_{-\infty}^{\infty} df S_x(f) = \int_0^{\infty} df S_x(f) + \int_0^{\infty} df S_x(-f) = \int_0^{\infty} df 2S_x(f)$$

Let us introduce  $\mathcal{S}_x(f) \equiv 2S_x(f) = \text{single-sided spectral density}$   
 (as opposed to  $S_x$  which is double-sided)

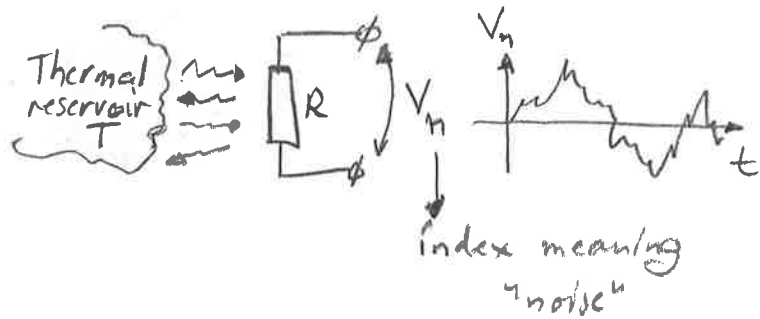
$$\overline{X^2} = \int_0^{\infty} df \cdot \mathcal{S}_x(f)$$

- This means that the average power of  $X$  is distributed over all frequencies. In some interval  $df$ ,  $\mathcal{S}_x(f) \cdot df$  represents the power obtained if a bandpass filter with bandwidth  $df$  is inserted before a power detector.
- Units of  $\mathcal{S}_x(f)$ : if  $X \equiv \text{voltage}$ , then  $V^2/\text{Hz}$ .

# Thermal noise in electrical circuits

Resistor at a finite temperature:

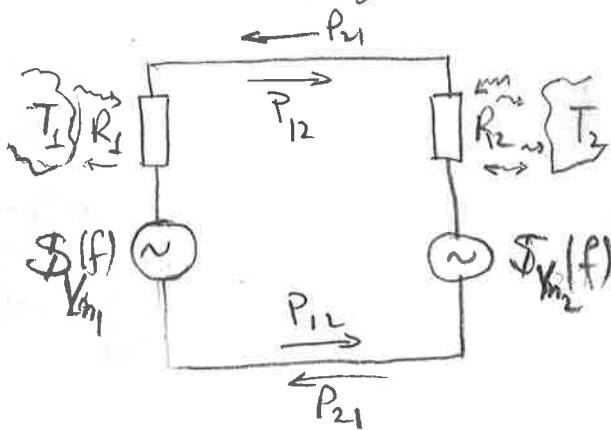
- Discovered by John Johnson (Bell Labs) in 1926
- Theory by Harry Nyquist (Bell Labs)



## Derivation of the Nyquist formula

(H. Nyquist, Thermal agitation of electric charge in conductors, Phys. Rev. 32, 110 (1928))

### 1. Estimates and general considerations



Calculate  $P_{12}$ :

$$V_2 = V_{n1} \frac{R_2}{R_1 + R_2}$$

$$P_{12} = \frac{V_2^2}{R_2} = V_{n1}^2 \cdot \frac{R_2}{(R_1 + R_2)^2}$$

Similarly for  $P_{21}$ :

$$P_{21} = V_{n2}^2 \cdot \frac{R_1}{(R_1 + R_2)^2}$$

1.a. Case  $T_1 = T_2 = T$

2nd law of Thermodynamics:  $\overline{P_{12}} = \overline{P_{21}}$ .

Otherwise by collecting the difference between them you would be able to extract work from a single reservoir at temperature  $T$ !

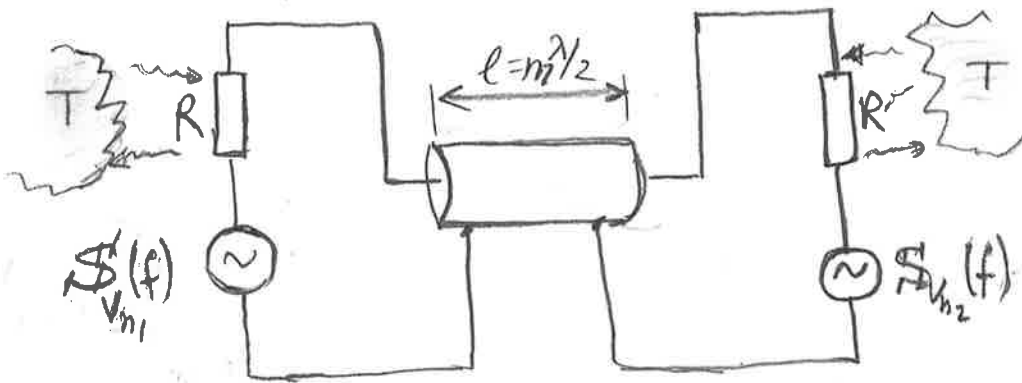
$$\overline{P_{12}} = \overline{P_{21}} \Rightarrow \frac{\overline{V_{n1}^2}}{R_1} = \frac{\overline{V_{n2}^2}}{R_2} = \text{const.} \Rightarrow \text{we expect } \overline{V_n^2} \sim R$$

2. b. Case  $T_1 \neq T_2$

We expect  $(\bar{P}_{12} - \bar{P}_{21}) \sim (T_1 - T_2)$ . This is just because usually energy  $\sim k_B T$  / Thermodynamics

Make  $T_2 = 0K$   
 $P_{21} = 0$  }  $\Rightarrow \bar{P}_{12} \sim T_1 \Rightarrow \overline{V_n^2} \sim T$

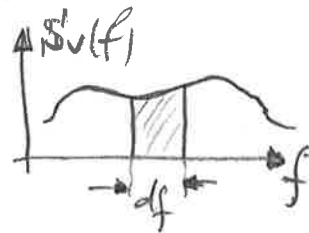
2. A more rigorous derivation



This time  $R_1 = R_2 = R$  but the connection is done through a TL (Transmission Line).

Consider a frequency interval  $df$ .

We want to find  $S_V(f)$  for a bandwidth  $df$  around  $f$ .



$v =$  speed of light in the TL

How many modes  $dm$  of the TL are in  $df$ ?

$l = m \frac{\lambda}{2} = m \frac{v}{2f} \Rightarrow f = m \frac{v}{2l} \Rightarrow dm = \frac{2l}{v} df$

(a)  $dE = dm \cdot \frac{hf}{e^{\frac{hf}{k_B T}} - 1} = \frac{2l}{v} \cdot \frac{hf}{e^{\frac{hf}{k_B T}} - 1} df =$  energy contained in the TL in the frequency interval  $df$

Planck's energy per mode

On the other hand, from  $\bar{P}_{12} = \overline{V_n^2} \frac{R}{(R+R)^2} = \frac{\overline{V_n^2}}{4R} = \bar{P}_{21}$ ,  $\bar{P} = \frac{\overline{V_n^2}}{4R}$

we have

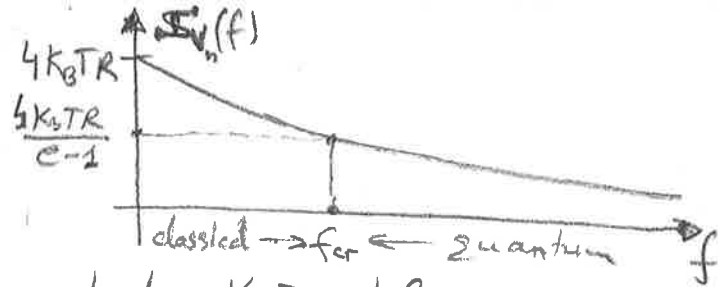
$d\bar{P} = \frac{1}{4R} S_V(f) \cdot df$

(b) therefore  $dE = 2 \cdot \frac{2l}{v} \cdot d\bar{P} = 2 \frac{l}{v} \cdot \frac{S_V(f)}{4R} \cdot df$

2 sources inject this power in TL  
 time of propagation in TL

Now combine (a) and (b) to find  $S_V(f)$ :

$$\Rightarrow S_{V_n}(f) = \frac{4hfR}{e^{hf/k_B T} - 1}$$



• Classical limit

If the temperature is high,  $k_B T \gg hf \Rightarrow$

$$S_{V_n}(f) \approx 4k_B T R$$

This could have been obtained also by applying the equipartition theorem:

$$dE = \frac{2\ell}{v} \cdot 2 \cdot \frac{k_B T}{2} \cdot df$$

$$\text{and } dE = \frac{2\ell}{v} \frac{S_{V_n}(f)}{4R} df$$

two degrees of freedom in the TL;  $\vec{E}$  and  $\vec{B}$   $\rightarrow \frac{k_B T}{2}$  per degree of freedom.

$$\Rightarrow S_{V_n}(f) \approx 4k_B T R$$

• Cross-over quantum-classical

From the experiment we can define  $f_{cr} = \frac{k_B T}{h}$

$$S_{V_n}(f) = \frac{4hfR}{e^{hf/k_B T} - 1} \approx 4hfR \cdot \frac{1}{1 + \frac{hf}{k_B T} + \frac{1}{2} \left( \frac{hf}{k_B T} \right)^2 + \dots}$$

$$\approx 4k_B T R \left( 1 - \frac{hf}{2k_B T} \right)$$

$$f_{cr} /_{T=300K} = \frac{1.38 \cdot 10^{-23} \text{ J/K} \cdot 300K}{6.62 \cdot 10^{-34} \text{ J}\cdot\text{s}} = 6.2 \text{ THz}$$

$$f_{cr} /_{T=10mK} = \frac{1.38 \cdot 10^{-23} \text{ J/K} \cdot 10^{-2} K}{6.62 \cdot 10^{-34} \text{ J}\cdot\text{s}} = 0.2 \text{ GHz} = 200 \text{ MHz}$$

- Thermal noise appears from the motion of electrons in the resistor. But where is the electron charge  $e$  in our final equations?!? It's in fact "hidden" in the microscopic models that we might use for  $R$  (for example the Drude model,

$$\sigma = \frac{1}{\rho} = \frac{ne^2 \tau}{m})$$

## Discussion:

• How much power it emits?  $\bar{P}_{tot} = \int_0^{\infty} df \frac{1}{R} S_{V_n}(f)$

$$= \frac{4(k_B T)^2}{h} \int_0^{\infty} \frac{u du}{e^u - 1} = \frac{2\pi^2}{3} \frac{(k_B T)^2}{h}$$

$$u = \frac{hf}{k_B T}$$

- Does not depend on  $R$ !

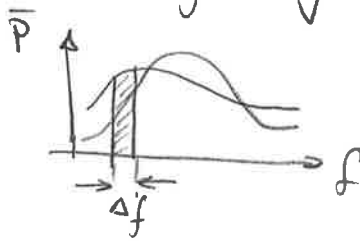
- Let's calculate it:  $T = 300K$

$$\bar{P}_{tot} = \frac{2\pi^2}{3} \cdot \frac{(1.38 \cdot 10^{-23} \cdot 300)^2}{6.62 \cdot 10^{-34}} \frac{1}{s} = 172 nW$$

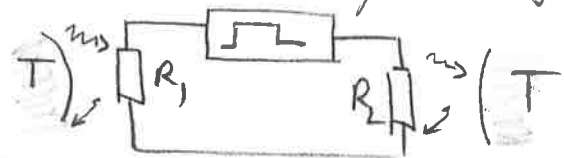
very small!

• Note that the power dissipated in the frequency interval  $df$   $\left\{ \frac{S_{V_n}(f)}{R} df \right\} = \text{independent of } R!$

Why it is so? It is a consequence of argument 1a in the beginning. Had it been otherwise,



it would be possible to create work from a single reservoir by using 2 resistors and a bandpass filter  $\Delta f$ .



This would contradict the 2<sup>nd</sup> law of thermodynamics (perpetuum mobile of 2<sup>nd</sup> kind)

• A more general statement: the fluctuation-dissipation theorem.

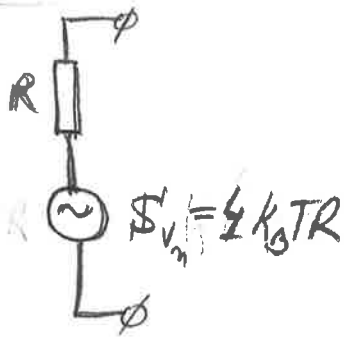
- in order for an object to emit radiation (due to fluctuations) it must also dissipate! Not restricted to electric circuits! E.g. in nanomechanics  $S_{FF}(\omega) = 2m k_B T \cdot \delta$

Example: A non-dissipative circuit element (ideal capacitor, inductance) will not emit radiation. There is no fluctuating voltage across an ideal capacitor! The ideal capacitor only stores energy. Also if you have an  $I-V$  characteristic and define  $dV/dI$  it does not mean that there will be accompanying fluctuations!

# Circuit equivalent representation

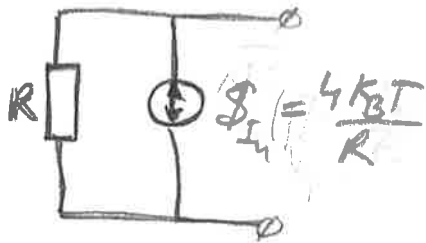
Consider from now on the classical limit,  $S_{V_n}(f) = 4k_BTR$ , we can obtain some useful and simple results.

## Thevenin



$\text{Ⓧ}$  = ideal voltage source (zero <sup>internal</sup> resistance) with fluctuating voltage

## Norton equivalent



$\text{Ⓧ}$  = ideal current source (∞ <sup>internal</sup> resistance) with fluctuating current

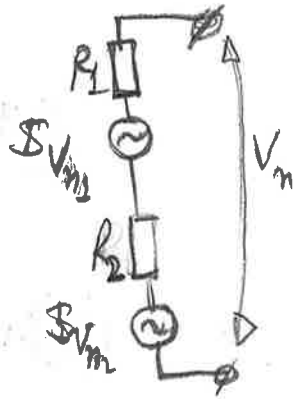
Quiz: why? convince yourself that this is the case

## Does it work?

Example: Resistors in series:

$$S_{V_{n1}} = 4k_BTR_1$$

$$S_{V_{n2}} = 4k_BTR_2$$



$S_{V_n} = S_{V_{n1}} + S_{V_{n2}}$  ← these fluctuations are uncorrelated!  
 so  $S_{V_n} = 4k_B T (R_1 + R_2)$   
 $R = R_1 + R_2 = \text{series}$

Resistors in parallel:

$$S_{I_{n1}} = \frac{4k_B T}{R_1}$$

$$S_{I_{n2}} = \frac{4k_B T}{R_2}$$



$S_{I_n} = S_{I_{n1}} + S_{I_{n2}}$  ← again uncorrelated fluctuations of current

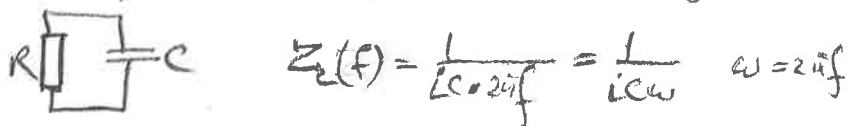
$$R_{||} = \frac{R_1 R_2}{R_1 + R_2}$$

So  $S_{I_n} = \frac{4k_B T}{R_{||}}$



o Coupling of a dissipative element to a non-dissipative one.

What happens for example when the fluctuations of the resistor charge and discharge a capacitor?



Norton



$$\frac{V_c(f)}{Z_c(f)} + \frac{V_c(f)}{R} = I_n(f) \text{ therefore}$$

$$S_{I_n} = \frac{4k_B T}{R}$$

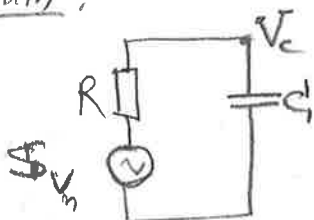
$$V_c(f) = \frac{I_n(f)}{\frac{1}{R} + iC \cdot 2\pi f} = \frac{I_n(f) \cdot R}{1 + 2\pi i f R C}$$

So we obtain

$$S_{V_c}(f) = \lim_{T \rightarrow \infty} \frac{|V_c(f)|^2}{T} = \frac{R^2}{|1 + 2\pi i f R C|^2} \cdot \lim_{T \rightarrow \infty} \frac{|I_n(f)|^2}{T} = \frac{R^2}{1 + (2\pi R C)^2 f^2} \cdot S_{I_n}$$

$$S_{V_c}(f) = \frac{4k_B T R}{1 + (2\pi R C)^2 f^2}$$

Thevenin:



$$V_c(f) = V_n(f) \cdot \frac{Z_c(f)}{R + Z_c(f)} \text{ therefore } V_c(f) = \frac{V_n(f)}{1 + 2\pi R C f}$$

$$S_{V_c}(f) = S_{V_n} \cdot \frac{1}{1 + (2\pi R C)^2 f^2} = \frac{4k_B T R}{1 + (2\pi R C)^2 f^2}$$

Same result! So per bandwidth the capacitor experiences

What is the total  $\overline{V_c^2}$  (over all frequencies)?

$$S_{V_c}(f) = \frac{4k_B T R}{1 + (2\pi R C)^2 f^2}$$

$$\overline{V_c^2}_{(tot)} = \int_0^{\infty} S_{V_c}(f) \cdot df = \frac{4k_B T}{2\pi C} \cdot \int_0^{\infty} df \frac{2\pi R C}{1 + (2\pi R C f)^2} = \frac{4k_B T}{2\pi C} \int_0^{\infty} dy \frac{1}{1 + y^2}$$

$$y = 2\pi R C f \quad = \frac{4k_B T}{2\pi C} \cdot \arctan y \Big|_0^{\infty}$$

$$= \frac{4k_B T}{2\pi C} \cdot \frac{\pi}{2} = \frac{k_B T}{C}$$

$$\text{So } \left\{ \overline{V_c^2}_{(tot)} = \frac{k_B T}{C} \right\}$$

This is called K-T-over-C-noise.

- Note that it does not depend on R! Why? - Noise increases with R but bandwidth decreases as  $\frac{1}{R}$ , so they compensate exactly.

- Suppose we want to increase the speed of a circuit (say a switch, AD converter, sampler, etc.) by making C smaller (smaller RC constant).

Then  $\overline{V_c^2}_{(tot)}$  will increase! The circuit will be more noisy!

- We can derive the formula above from the equipartition theorem

$$\frac{1}{2} C \overline{V_c^2} = \frac{1}{2} k_B T \leftarrow \text{There is only 1 degree of freedom in a capacitor.}$$

- Take-home conclusion: dissipation (by resistive elements) produces noise. The noise tends to "bury" any quantum effects, making them useless.

Two ways out of it:

1) Increase the frequency. This can be done in some systems (optics) but it is not always possible with microwaves and rf.

2) Use zero-resistance components. Is this possible?

Yes. We will see in the next lecture.

# References

- Hermann A. Haus - Electromagnetic Noise and Quantum Optics
- Ali Hajimiri - Analog Circuit Design (lectures at Caltech)
- A.M. Zagorkin - Quantum Engineering

Highly advanced:

- C. Gardiner - Quantum Noise
- M. Cattaneo, GSP - arXiv:2103.16946