Tutorial notes 6

Quantum Gates

(1) Given the definitions of single qubit operations, show the following properties
$(a)\, XYX = -Y \quad (b)\, HXH = Z \quad (c)\, HYH = -Y \quad (d)\, HZH = X$

(2) Suppose that $|\varphi_{\text{in}}\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$. What is the final state $|\varphi_{\text{out}}\rangle$ in each of the following two cases?

$$|\varphi_{\text{in}}\rangle \;—\boxed{X}—\boxed{H}—\boxed{Z}—\; |\varphi_{\text{out}}\rangle \qquad\qquad |\varphi_{\text{in}}\rangle \;—\boxed{Z}—\boxed{H}—\boxed{X}—\; |\varphi_{\text{out}}\rangle$$

(3) Consider a qubit in the $|0\rangle$ state entering a Hadamard gate. Then, the Hadamard output is fed into a CNOT gate as the *control* qubit while the *target* in initiated in the $|0\rangle$.

(a) Draw a diagram of the above circuit, and write down its final output state.

(b) Now perform separately the following 2-qubit operations on the circuit's output $I \otimes X$, $I \otimes Y$, $I \otimes Z$. What is the output state in each case?

(4) The Hamiltonian of the **iSWAP** gate is defined as $H = \frac{\hbar g}{2}(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y)$. Define a unitary evolution as $U = e^{-iHt/\hbar}$

$= e^{-i\frac{g}{2}t(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y)}$

(a) Derive the matrix representation of this transformation.
Hint: $e^{i\theta A} = \cos\theta I + i\sin\theta A$. $e^{A+B} = e^A e^B e^{-1/2[A,B]}$. where $A, B \in L(\mathcal{H})$.

$t_g = \frac{\pi}{2g}$

(b) Describe the action of the $\sqrt{\textbf{iSWAP}}$ on the state $|0\rangle \otimes |1\rangle$.

$U(t)|\varphi\rangle(t)$

$gt \rightarrow t = \frac{\pi}{2g}$ $\quad t_g = \frac{\pi}{4g}$

$$\text{iswap} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos gt & -i\sin gt & 0 \\ 0 & -i\sin gt & \cos gt & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(1) $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$; $Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$; $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$\text{iswap} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

→ Decomposition of matrices ⇒ Change of basis.

(2) $ZHX|\varphi_{in}\rangle = ZHX(\alpha|0\rangle + \beta|1\rangle)$

$XHZ|\varphi_{in}\rangle = XHZ(\alpha|0\rangle + \beta|1\rangle)$

$Z|0\rangle = |0\rangle$; $Z|1\rangle = -|1\rangle$

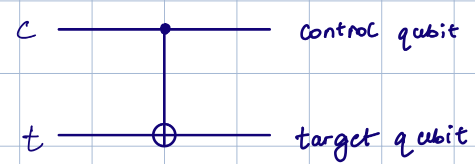
$X|0\rangle = |1\rangle$; $X|1\rangle = |0\rangle$

$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(3) $\text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$



Combined system: $t \otimes c$

$$|0\rangle \otimes |1\rangle = |01\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$|00\rangle =$

$|10\rangle =$

$|11\rangle =$

Ex: $\begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \cdot x \\ \begin{bmatrix} a \\ b \end{bmatrix} \cdot y \end{bmatrix}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot x & \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot y \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z & \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot w \end{bmatrix}$$

(i) $I \otimes X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

(4) $e^{i\theta \hat{A}} = \cos\theta \cdot \hat{I} + i\sin\theta \cdot \hat{A}$ $\qquad e^{i\theta} = \cos\theta + i\sin\theta$

$e^{A+B} = e^{A} e^{B} e^{-\frac{1}{2[A,B]}}$ $\qquad \sigma_y \sigma_x = I$

$$\hat{U}_{qt}(t) = e^{-i\frac{g}{2}(\underbrace{\sigma_x \otimes \sigma_x}_{X := \sigma_x \otimes \sigma_x} + \underbrace{\sigma_y \otimes \sigma_y}_{Y := \sigma_y \otimes \sigma_y})t}$$

$$= e^{-i\frac{g}{2}(\hat{X} + \hat{Y})}$$

$$= e^{-i\underbrace{g/2}_{\theta}\hat{X}} \cdot e^{-i\underbrace{g/2}_{\theta}\hat{Y}} \, e^{\cancel{-\frac{1}{2[A,B]}}} \; \hat{I}$$

$X = \sigma_x \otimes \sigma_x = \Big[\quad \Big]_{4\times 4}$

$Y = \sigma_y \times \sigma_y = \Big[\quad \Big]_{4\times 4}$

$e^{X+Y} = e^{X} \cdot e^{Y}$

$[X, Y] = XY - YX = \bar{0}$