Lecture notes exhaustible resources

Matti Liski

May 18, 2022[*]

Abstract

These notes present a simple model to understand the basic economics of scarcity.

[*]Edited by by Iivo Vehviläinen in 2020.

1 Introduction

The purpose of this short note is to support the lecture on exhaustible resources. We will analyze this setting graphically in the class.

2 The model

2.1 The arbitrage condition

Suppose interest rate $r > 0$ is constant over time, and that there is a constant cost $c > 0$ per unit of resource produced and sold to the consumers. The total available resource is of size $S_0 > 0$. That is, over time consumption cannot exceed S_0.

If a unit of the resource is sold today at time t, the seller receives:

$$P_t - c.$$

Alternatively, the seller can postpone the sale by one period. We let $\Delta > 0$ to denote how many units of time is "one period".[1] Thus, when the sale is postponed by one period, it takes place at time $t + \Delta$ where the price is $P_{t+\Delta}$. The unit cost c remains the same at the next period; it was assumed to be a constant independent of time. The final element relevant considering the gains from the delay of sales is discounting. We assumed above that $r > 0$ is the interest rate, so $e^{-\Delta r}$ is the discount factor for waiting $\Delta > 0$ units of time.[2] Thus, if the producer sells next period, after $\Delta > 0$ units of time, value of the sale when discounted to the present is

$$[P_{t+\Delta} - c]e^{-\Delta r}$$

In equilibrium, the sellers should be indifferent between selling today or in the next period; otherwise, they would all sell today or save the resource.[3] The indifference requires

$$\begin{aligned} P_t - c &= [P_{t+\Delta} - c]e^{-\Delta r} \\ &\approx [P_{t+\Delta} - c](1 - \Delta r) \end{aligned}$$

[1] This proves useful because working with very small $\Delta > 0$ allows us to use differential methods.

[2] You may be more used to a discount factor of the form $(1 + \overline{r})^{-t}$ where $t = 1, 2, 3..$ denotes years (for example) and $\overline{r}$ is the *annual discount rate*. The two discount rates are linked by the equation

$$e^{-\Delta r} = \frac{1}{(1 + \overline{r})^{\Delta}}$$

so we can equivalently work with $e^{-\Delta r}$ which is more convenient analytically.

[3] Of course, this could be an equilibrium outcome as well but this requires very special assumptions regarding the value of consumption, as will be seen. Typically, we are interesting in situation where consumption is valued in both periods.

The second line follows since $e^{-\Delta r} \approx (1 - \Delta r)$ for small period length.[4] Then, if we rearrange the indifference condition, we obtain

$$\frac{P_{t+\Delta} - P_t}{\Delta} = r(P_{t+\Delta} - c)$$
$$\Rightarrow$$
$$\lim_{\Delta \to 0} = \frac{dP_t}{dt} = r(P_t - c)$$

The second line is just the definition of a derivative. The last equation tells how the price must change over time for the indifference to hold. It is a differential equation which has the following solution:

$$P_t = c + [P_0 - c]e^{rt}, \tag{1}$$

where P_0 is the initial value (value of price at time zero). If $c = 0$, then the price must grow at the rate of interest: the resource left in the ground appreciates exactly at the same rate as the money in the bank account had the seller sold at time $t = 0$. If $c > 0$, then the same reasoning is applied to the net revenue from sales.

2.2 The terminal price condition

We know that the equilibrium price must satisfy (1). But two questions remain. First, how high will the price grow? Second, at what level will the price path start? We answer the first question first by defining the "terminal price condition". Effectively, the price cannot become higher than the consumers' willingness to pay. Recall that demand is a relationship between quantities and prices $Q_t = D(P_t)$ where Q_t is demand (=consumption) at time t, and $D'(P_t) < 0$. Choke price, $\overline{P}$, is defined as the price at which the demand chokes off: $D(\overline{P}) = 0$ for all $P \geq \overline{P}$. Thus, no demand for prices higher than $\overline{P}$. We assume that this choke price is finite.

Let T be the time it takes for price following (1) to reach $\overline{P}$, that is,

$$P_T = c + [P_0 - c]e^{rT} = \overline{P}. \tag{2}$$

This is the terminal price condition. It pins down the end of the price path.

2.3 The exhaustion condition

Now, we answer the second question, that is, we pin down the level at which the price path will start. Along the way, we end up determining how long the consumption path lasts, that, the precise value for T. To see why the initial value of the price matters, consider (2) and note that if P_0 is high, we will reach $\overline{P}$ sooner than if P_0 is low. That is, for low starting price, it takes a long time to reach the final price. Then, this means that the price at each time point t is lower than if we start with a high initial price. Since the consumption at each time is higher, the lower is the price, we see that the initial price

[4] This follows by a method called linear approximation of a nonlinear function.

influence the total amount of consumption. The only way to pin down the price level is to make sure that the total consumption over time equals the amount of the resource available.

$$\int_0^T Q_t dt = S_0 \tag{3}$$

This is the exhaustion condition. Since we know the demand, we can write

$$\int_0^T D(P_t) dt = S_0 \tag{4}$$

This may look difficult, but the equilibrium is conceptually very simple. We have three unknowns: P_0, P_T, T. There are three equations for finding these: (1), (2), and (3).