Mathematics for Economists: Lecture 1

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Welcome to the course

- Course logistics
  - Lectures Mon, Tue at 13:15 - 14:45
  - Review sessions with Amin Mohazab Thu 10:15 - 11:45
  - This week: Tue at 13:15 - 14:45, Thu 10:15 - 11:45, first review session Fri 10:15 -11:45
  - Weekly Problem Sets to be returned via MyCourses on specified due date (not first week)
  - 20% of the grade based on problem sets, 80% on final exam
  - To succeed in the course, you should attempt all problem sets
  - Exam June 1, 9:00-12:00

Contents I

- Course contents: Part I
  - Lectures 1-2 Introduction and Applications of Linear Algebra Readings: Synopsis, Section 1, material for week 1, S&B: Chapter 6-11, 13.
  - Lectures 3-4: Multivariable Calculus Readings: Synopsis, Section 2, material for week 2, S&B: Chapters 14, 15.
  - Lecture 5: Unconstrained Optimization, Convexity and Concavity Readings: Synopsis, Section 3, material for week 3, S&B: Chapters 16, 17, 21.

Contents II

- Course contents: Part II
  - Lecture 7-8: Constrained Optimization Readings: Synopsis, Sections 4-5, material for week 4, S&B Chapters 18, 19
  - Lecture 9-10: Economic Applications of Constrained Optimization Readings: Synopsis, Sections 4-5: material for week 5, S&B:Chapters 20, 22
  - Lectures 11-12: Linear Dynamical Systems
    Readings: Synopsis, Sections 6: material for week 6, S&B: 23, 25.1, 25.2

Mathematics and Economic Models

- In Lionel Robbins' definition, economics studies the allocation of scarce resources amongst competing ends
- We seek a mathematical formulation for such problems
  - Competing ends captured by an objective function
  - Scarcity captured by feasible set
  - Economic process captured by either a constrained optimization problem of an economic agent or an equilibrium problem where many agents optimize simultaneously
  - Handout on *Setting the Scene* contains more details on this.
- Mathematical requirements for the theory
  - Choices over vectors (e.g. consumption bundles in the budget set or choice of inputs for a producer) $->$ Vectors and linear algebra
  - Objective functions often not linear $->$ Multivariable calculus
  - Environment for choices may evolve over time $->$ Dynamical systems

Linear economic models

- The key object of study in linear algebra are *linear equations*
- A linear equation has the form

  $$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b,$$

  where
  - $a_1, a_2, \ldots, a_n, b$ are fixed real numbers (parameters)
  - $x_1, x_2, \ldots, x_n$ are real valued variables
- A system of linear equations is a collection of such equations that hold simultaneously

Examples of systems

A system of two linear equations in two unknowns:

$$2x_1 + 3x_2 = 7 \quad (1)$$

$$2x_1 + x_2 = 4 \quad (2)$$

A system of two non-linear equations in two unknowns:

$$2x_1 x_2 + 3x_2 = 7 \quad (3)$$

$$2x_1^2 + x_2 = 4 \quad (4)$$

Market Equilibrium for Two Goods

- Goal: Determine competitive equilibrium prices and quantities for two goods $i \in \{1, 2\}$.
- Demand Q_i^d depends on the prices of the two goods P_1 and P_2, on disposable income Y and on other factors K_i as follows:

$$Q_1^d = K_1 P_1^{\alpha_{11}} P_2^{\alpha_{12}} Y^{\beta_1},$$

$$Q_2^d = K_1 P_1^{\alpha_{21}} P_2^{\alpha_{22}} Y^{\beta_2}.$$

- How would you interpret the parameters α_{ij} ja β_i?
- Think back to Principles 1 and elasticities

Market Equilibrium for Two Goods

- Y and K_i are the exogenous variables (i.e. ones not determined in the model).
- Supply functions Q_i^s for the two products are assumed to take the form:

$$Q_1^s = M_1 P_1^{\gamma_1},$$

$$Q_2^s = M_2 P_2^{\gamma_2}.$$

- Again, we take the variables M_i to be exogenous to the model.
- In equilibrium, supply equals demand so that

$$Q_1^d = Q_1^s,$$

and

$$Q_i^d = Q_i^s.$$

- Six equations for six endogenous variables $(Q_i^s, Q_i^d, P_i)_{i=1,2}$, but not linear

Market Equilibrium for Two Goods

- But a change of variables helps: define the following new variables for $i \in \{1, 2\}$:

$$q_i^d = \ln Q_i^d, \; q_i^s = \ln Q_i^s \; p_i = \ln P_i, y = \ln Y, \; m_i = \ln M_i, \; k_i = \ln K_i.$$

- By taking logarithms on both sides of each equation, we can write the six equations for $i \in \{1, 2\}$:

$$q_i^d = k_i + \alpha_{ii} p_i + \alpha_{ij} p_j + \beta_i y,$$

$$q_i^s = m_i + \gamma_i p_i,$$

$$q_i^s = q_i^d.$$

Market Equilibrium for Two Goods

- By the third equation, $q_i^d = q_i^s$ for $i \in \{1, 2\}$, and therefore the right hand sides in the first and the second equations are equalized:

$$k_i + \alpha_{ii} p_i + \alpha_{ij} p_j + \beta_i y = m_i + \gamma_i p_i, \;\; i \in \{1, 2\}.$$

- The only remaining endogenous variables are: p_1 ja p_2.
- Let's write the exogenous variables on the right-hand side and the endogenous variables on the left-hand side:

$$\begin{array}{ccc} (\alpha_{11} - \gamma_1)\, p_1 & +\alpha_{12} p_2 & = m_1 - k_1 - \beta_1 y, \\ \alpha_{21} p_1 & (\alpha_{22} - \gamma_2)\, p_2 & = m_2 - k_2 - \beta_2 y. \end{array}$$

Market Equilibrium for Two Goods

- Let's solve for p_1 from the top equation:

$$p_1 = \frac{m_1 - k_1 - \beta_1 y - \alpha_{12} p_2}{(\alpha_{11} - \gamma_1)}.$$

- Substituting into the second equation gives:

$$\begin{aligned} p_2 &= \frac{m_2 - k_2 - \beta_2 y - \alpha_{21} p_1}{(\alpha_{22} - \gamma_2)} \\ &= \frac{m_2 - k_2 - \beta_2 y - \alpha_{21} \frac{m_1 - k_1 - \beta_1 y - \alpha_{12} p_2}{(\alpha_{11} - \gamma_1)}}{(\alpha_{22} - \gamma_2)}. \end{aligned}$$

- Multiplying both sides by $(\alpha_{22} - \gamma_2)(\alpha_{11} - \gamma_1)$ gives:

$$(\alpha_{22} - \gamma_2)(\alpha_{11} - \gamma_1) p_2 = (\alpha_{11} - \gamma_1)(m_2 - k_2 - \beta_2 y)$$
$$-\alpha_{21}(m_1 - k_1 - \beta_1 y) + \alpha_{12}\alpha_{21} p_2,$$

Market Equilibrium for Two Goods

- Solving for p_2,

$$p_2 = \frac{(\alpha_{11} - \gamma_1)(m_2 - k_2 - \beta_2 y) - \alpha_{21}(m_1 - k_1 - \beta_1 y)}{(\alpha_{22} - \gamma_2)(\alpha_{11} - \gamma_1) - \alpha_{12}\alpha_{21}}.$$

- And substituting back gives:

$$\begin{aligned} p_1 &= \frac{m_1 - k_1 - \beta_1 y - \alpha_{12}\frac{(\alpha_{11}-\gamma_1)(m_2-k_2-\beta_2 y)-\alpha_{21}(m_1-k_1-\beta_1 y)}{(\alpha_{22}-\gamma_2)(\alpha_{11}-\gamma_1)-\alpha_{12}\alpha_{21}}}{(\alpha_{11} - \gamma_1)} \\ &= \frac{(\alpha_{22} - \gamma_2)(m_1 - k_1 - \beta_1 y) - \alpha_{12}(m_2 - k_2 - \beta_2 y)}{((\alpha_{22} - \gamma_2)(\alpha_{11} - \gamma_1) - \alpha_{12}\alpha_{21})}. \end{aligned}$$

Market Equilibrium for Two Goods

- The (logarithmic) equilibrium quantities are solved most easily from the supply curves.
- Finally P_i, Q_i are solved by exponentiating p_i, q_i.
- Exercise: Can you see how an improvement in the production technology for good 2 changes the equilibrium?
- **Lessons from the example**
- A nonlinear model can sometimes be transformed to a linear model (logarithmic transforms are particularly useful)
- Solving by substitution is clumsy and prone to errors.
- Gaussian elimination (familiar from Matrix Algebra) is a systematic representation of this process .

Gaussian Elimination

- Consider the following system of linear equations:

$$\begin{aligned} 2x_1 - x_2 &= 0 \\ -x_1 + 2x_2 - x_3 &= 0 \\ -x_2 + 2x_3 - x_4 &= 0 \\ -x_3 + 2x_4 &= 5 \end{aligned}$$

- Write this in matrix form:

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 5 \end{pmatrix}.$$

Gaussian Elimination

- The augmented matrix corresponding to this system is:

$$\left(\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{array}\right).$$

- Step 1: Eliminate all terms a_{j1} below the first *pivot* a_{11} (i.e. first non-zero element in column 1). In this case, just multiply the first row by $\frac{1}{2}$ and add to second row.

$$\left(\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{array}\right)$$

Gaussian Elimination

- Step 2: Eliminate all terms blow the second pivot: add the second row multiplied by $\frac{2}{3}$ to the third row.

$$\left(\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & -1 & 0 \\ 0 & 0 & -1 & 2 & 5 \end{array}\right)$$

Gaussian Elimination

- Step 3: Eliminate the term below the third pivot: add row 3 multiplied by $\frac{3}{4}$ to row 4.

$$\left(\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & -1 & 0 \\ 0 & 0 & 0 & \frac{5}{4} & 5 \end{array}\right)$$

- Step 4: At this point, the matrix is in *row echelon form*, i.e. each row starts with more zeros than the previous row.
- Last row reads: $\frac{5}{4}x_4 = 5$ or $x_4 = 4$. You can substitute this back to row 3 that says $\frac{4}{3}x_3 - x_4 = 0$ to get $x_3 = 3$. From row 2, $x_2 = 2$ and from row 1, $x_1 = 1$.
- In Gauss-Jordan elimination, one eliminates all elements above and below the pivot in these steps.
- This requires more eliminations, but avoids the substitutions in the last step

Gaussian Elimination: Second Example

- Consider another example:

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 + x_2 + x_3 &= 2 \\ x_2 + x_3 &= 1 \end{aligned}$$

- In matrix form:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{array}\right)$$

Gaussian Elimination: Second Example

- Step 1:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right)$$

- Notice that now the second pivot is in row 3. Swapping rows 2 and 3 does not change the solution to an equation. So Step 2: Swap rows 2 & 3:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

- Now you can read $x_3 = 1$, $x_2 = 0$, and $x_1 = 1$.

Gaussian Elimination: General Principles

- Three *elementary row operations* that leave the solutions to systems of equations unchanged:
  1. Multiplying a row by a real number
  2. Adding rows to other rows
  3. Swapping rows
- Every matrix can be transformed to its row echelon form by elementary row operations.
- The *rank* of a matrix is the number of non-zero rows in its row echelon form.
- A linear system of equations $\boldsymbol{Ax} = \boldsymbol{b}$ has a solution if and only if the rank of the coefficient matrix $\boldsymbol{A}$ is equal to the rank of the augmented matrix $(\boldsymbol{A}|\boldsymbol{b})$.
- If rank $(\boldsymbol{A})$ = rank $(\boldsymbol{A}|\boldsymbol{b})$ = n, the solution is unique, if rank $(\boldsymbol{A})$ = rank $(\boldsymbol{A}|\boldsymbol{b}) < n$, then the system has infinitely many solutions.

Next Lecture

- A brief review of some of the main concepts of Matrix Algebra or Linear Algebra
- Input-output models of economic production
- Linear models of exchange