The Norm and Inner Product

If the inner product is defined as:

$$\boldsymbol{x} \cdot \boldsymbol{y} = \sum_{i=1}^{n} x_i y_i,$$

the norm can be defined un terms of the inner product:

$$\|\boldsymbol{x}\|^2 := \boldsymbol{x} \cdot \boldsymbol{x}.$$

The projection of a vector $\boldsymbol{y}$ on $\boldsymbol{x}$ is defined as the point $t^*\boldsymbol{x}$ on the line $t\boldsymbol{x}$ for $t \in \mathbb{R}$ such that

$$(\boldsymbol{y} - t^*\boldsymbol{x}) \cdot \boldsymbol{x} = 0.$$

This gives an explicit formula for t^*:

$$t^* = \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{\|\boldsymbol{x}\|^2}.$$

Hence the projection $P_{\boldsymbol{x}}(\boldsymbol{y})$ is given by:

$$P_{\boldsymbol{x}}(\boldsymbol{y}) = \boldsymbol{x}\frac{\boldsymbol{x} \cdot \boldsymbol{y}}{\|\boldsymbol{x}\|^2}.$$

By basic trigonometry, the angle θ between $\boldsymbol{x}$ and $\boldsymbol{y}$ satisfies:

$$\cos(\theta) = \frac{\|P_{\boldsymbol{x}}(\boldsymbol{y})\|}{\|\boldsymbol{y}\|} = \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{\|\boldsymbol{x}\|\|\boldsymbol{y}\|}.$$

Since $-1 \leq \cos(\theta) \leq 1$ for all θ, we get Cauchy's inequality for all vectors $\boldsymbol{x}, \boldsymbol{y}$:

$$|\boldsymbol{x} \cdot \boldsymbol{y}| \leq \|\boldsymbol{x}\|\|\boldsymbol{y}\|.$$