Mathematics for Economists
ECON-C1000
Spring 2022
Juuso Välimäki
juuso.valimaki@aalto.fi

Problem Set 1 : Due April 28, 2022

1. Solve the following linear systems by Gaussian Elimination and by Cramer's Rule.

   (a)

$$\begin{aligned} 3x_1 - x_3 &= 0 \\ -x_1 + 2x_3 &= 0 \\ -x_1 + 4x_2 &= 2 \end{aligned}$$

   (b)

$$\begin{aligned} x_1 - x_3 - x_4 &= 0 \\ -x_1 + x_2 + 2x_3 &= 0 \\ -x_1 + 4x_2 &= 2 \\ x_2 + x_3 - x_4 &= 1 \end{aligned}$$

2. Prove or disprove the following claims:

   (a) For an arbitrary matrix $\boldsymbol{A}$, the matrix $\boldsymbol{B} = \boldsymbol{A}^\top \boldsymbol{A}$ (where $\boldsymbol{A}^\top$ is the transpose of $\boldsymbol{A}$) is symmetric, i.e. that for all elements of $\boldsymbol{B}$, $b_{ij} = b_{ji}$.

   (b) If the $n \times n$ matrix $\boldsymbol{A}$ does not have full rank and $\boldsymbol{B}$ is another $n \times n$ matrix, then the $n \times n$ matrix $\boldsymbol{C} = \boldsymbol{A}\boldsymbol{B}$ does not have full rank.

   (c) If $\boldsymbol{x}$ solves $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, then $\boldsymbol{x}^\top$ solves $\boldsymbol{x}^\top \boldsymbol{A} = \boldsymbol{b}^\top$.

3. Show that the following system has a unique solution when $\beta \neq \alpha$ and that the solution is positive when $\beta > \alpha > 0$ and $\gamma > 0$.

$$\begin{pmatrix} \beta & \alpha & \alpha \\ \alpha & \beta & \alpha \\ \alpha & \alpha & \beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \gamma \\ \gamma \\ \gamma \end{pmatrix}.$$

4. A stochastic matrix is a square matrix whose the elements in each column sum up to 1. In other words, if $\boldsymbol{A}$ is a stochastic $n \times n$ -matrix, then $\sum_{i=1}^{n} a_{ij} = 1$ for all j.

   (a) Show that the matrix

$$\boldsymbol{A} = \begin{pmatrix} 0.3 & 0.5 & 0.4 \\ 0.4 & 0.1 & 0.5 \\ 0.3 & 0.4 & 0.1 \end{pmatrix}$$

   has full rank, but the matrix

$$(\boldsymbol{I} - \boldsymbol{A}) = \begin{pmatrix} 0.7 & -0.5 & -0.4 \\ -0.4 & 0.9 & -0.5 \\ -0.3 & -0.4 & 0.9 \end{pmatrix},$$

   where $\boldsymbol{I}$ is the $n \times n$ identity matrix, does not have full rank.

   (b) Find a vector $\boldsymbol{x} \neq 0$ such that $(\boldsymbol{A} - \boldsymbol{I})\boldsymbol{x} = 0$.

   (c) Show that if $\boldsymbol{A}$ is any stochastic matrix, then $(\boldsymbol{I} - \boldsymbol{A})$ and $(\boldsymbol{A} - \boldsymbol{I})$ do not have full rank.

5. Consider a class of 30 students. A sociologist wants to understand the social hierarchy in the class and asks each student to endorse at least one other student in the class. Based on the responses she designs a popularity ranking for the students. We want to see how this can be done using the tools of linear models.

   (a) Form a matrix of endorsements as follows: Identify each row and each column in a 30×30 -matrix $\boldsymbol{A}$ with a student. The students endorsed by student j form the j^{th} column of the matrix as follows. If j endorses n_j other students, set $a_{ij} = \frac{1}{n_j}$ if j endorsed i and 0 otherwise (no self-endorsements). What can you say about $\sum_{i=1}^{30} a_{ij}$?

   (b) What is the interpretation of $y_i := \sum_{j=1}^{30} a_{ij}$? Is y_i a good measure for the popularity of the students?

   (c) Maybe endorsements from popular students are more important for the ranking. To capture this idea, consider a ranking vector $\boldsymbol{x} = (x_1, ..., x_30)$ for the students. Require that the ranking of each student i is the weighted sum endorsements a_{ij} wighted by the popularity ranking of the endorsing student j. Write the linear model capturing this idea as:

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{x}.$$

   Show that $(\boldsymbol{A} - \boldsymbol{I})$ does not have full rank. This means that there is a non-zero solution to the linear system.

(d) You can take it on faith (or find a proof using either Brouwer's fixed-point theorem or Farkas' lemma) that a strictly positive solution exists. Normalize the solution $\boldsymbol{x}$ so that $\sum_{i=1} 30x_i = 1$. Show by an example that there can be many such non-zero solutions if the students can be divided into cliques that do not endorse students in other cliques.