Mathematics for Economists

Instructor: Juuso Valimaki

Teacher Assistant: Amin Mohazab

amin.mohazabrahimzadeh@aalto.fi

Solutions to the problem set 1

Question 1:

a) We solve the following system of equations using Gaussian elimination and Cramer's rule:

$$\begin{pmatrix} 3 & 0 & -1 \\ -1 & 0 & 2 \\ -1 & 4 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

Using Gaussian elimination, we immediately realize we can not set x_2 coefficient in the third equation to zero because the coefficients of x_2 in other 2 equations are already zero. We try swapping the second and third rows, since it will have no effect on the results.

$$\begin{pmatrix} 3 & 0 & -1 \\ -1 & 4 & 0 \\ -1 & 0 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

We then form the augmented matrix:

$$\left[\begin{array}{ccc|c} 3 & 0 & -1 & 0 \\ -1 & 4 & 0 & 2 \\ -1 & 0 & 2 & 0 \end{array}\right]$$

using elementary row operations: (we use the first row to set the x_1 coefficients in second and third row to zero)

$$\left[\begin{array}{ccc|c} 3 & 0 & -1 & 0 \\ -1 & 4 & 0 & 2 \\ -1 & 0 & 2 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 3 & 0 & -1 & \\ 0 & 4 & -\frac{1}{3} & 0 \\ & & & 2 \\ 0 & 0 & \frac{5}{3} & 0 \end{array}\right]$$

$$x_3 = 0$$

$$4x_2 - \frac{1}{3}x_3 = 0 \rightarrow x_2 = \frac{1}{2}$$

$$3x_1 - x_3 = 0 \rightarrow x_1 = 0$$

Using Cramer's rule, we have:

$$x_1 = \frac{\det(A_x)}{\det(A)} = \frac{\det\left(\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 2 \\ 2 & 4 & 0 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 3 & 0 & -1 \\ -1 & 0 & 2 \\ -1 & 4 & 1 \end{bmatrix}\right)} = \frac{0}{-4(6+1)} = 0$$

$$x_2 = \frac{\det(A_y)}{\det(A)} = \frac{\det\left(\begin{bmatrix} 3 & 0 & -1 \\ -1 & 0 & 2 \\ -1 & 2 & 1 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 3 & 0 & -1 \\ -1 & 0 & 2 \\ -1 & 4 & 1 \end{bmatrix}\right)} = \frac{-2(6+1)}{-4(6+1)} = \frac{1}{2}$$

$$x_3 = \frac{\det(A_z)}{\det(A)} = \frac{\det\left(\begin{bmatrix} 3 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 4 & 2 \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 3 & 0 & -1 \\ -1 & 0 & 2 \\ -1 & 4 & 1 \end{bmatrix}\right)} = \frac{0}{-4(6+1)} = 0$$

b)

$$\begin{bmatrix} 1 & 0 & -1 & -1 \\ -1 & 1 & 2 & 0 \\ -1 & 4 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

we first follow the elementary row operations:

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ -1 & 1 & 2 & 0 & 0 \\ -1 & 4 & 0 & 0 & 2 \\ 0 & 1 & 1 & -1 & 1 \end{array}\right]$$

Subtracting the second row from the fourth row, we have:

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ -1 & 1 & 2 & 0 & 0 \\ -1 & 4 & 0 & 0 & 2 \\ 1 & 0 & -1 & -1 & 1 \end{array}\right]$$

In the next step, we subtract the first row from the last one:

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ -1 & 1 & 2 & 0 & 0 \\ -1 & 4 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

It seems like that the determinant of the matrix of the coefficients is zero and as the result, this system of equations does not have any answer. To be more clear, consider the fourth equation in the last matrix:

$$0x_1 + 0x_2 + 0x_3 + 0x_4 = 1$$

clearly, there are no values of independent variables (x_1, x_2, x_3, x_4) that can satisfy this equation.

We could also calculate the determinant of the matrix at the beginning and derive the same results. Although determinant of a 4 by 4 matrix is hard to derive, the coefficient matrix has many zero elements that can help us a lot. In the following, we will use the definition of sub-matrices to calculate determinant:

$$\det\left(\begin{bmatrix} 1 & 0 & -1 & -1 \\ -1 & 1 & 2 & 0 \\ -1 & 4 & 0 & 0 \\ 0 & 1 & 1 & -1 \end{bmatrix}\right) = (-1).\det\left(\begin{bmatrix} -1 & 1 & 2 \\ -1 & 4 & 0 \\ 0 & 1 & 1 \end{bmatrix}\right) -$$

$$0.\det\left(\begin{bmatrix} 1 & 0 & -1 \\ -1 & 4 & 0 \\ 0 & 1 & 1 \end{bmatrix}\right) + 0.\det\left(\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}\right) - (-1).\det\left(\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 2 \\ -1 & 4 & 0 \end{bmatrix}\right) =$$

$$(-1).\left((-1).\det\left(\begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix}\right) - (-1).\det\left(\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\right)\right) -$$

$$(-1).\left(1.\det\left(\begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix}\right) + (-1).\det\left(\begin{bmatrix} -1 & 1 \\ -1 & 4 \end{bmatrix}\right)\right) =$$

$$(-1)\big((-1).4 - (-1).(-1)\big) - (-1)\big(-8 + (-1)(-3)\big) =$$

$$(-1)(-5) - (-1)(-5) = 0$$

Note that we chose the last column of the matrix since it has two zero elements and it makes the calculation much simpler.

Moreover, it is not possible to use Cramer's rule in here because the determinant of the coefficient matrix (the denominator) is equal to zero.

Question 2:

a)

We can prove the following properties for the transpose of a matrix:

1) $(A^T)^T = A$
2) $(A+B)^T = A^T + B^T$
3) $(kA)^T = kA^T$
4) $(AB)^T = B^T A^T$ (to prove it just compare the (i,j) entries of $(AB)^T$ and $B^T A^T$)

using the properties 1 and 4:

$$B = (AA^T)^T = {A^T}^T A^T = AA^T$$

so matrix B is symmetric.

b)

It can be proved that for any n by n matrices A and B[1]:

[1] Please visit here for the proofs.

$$\det(AB) = \det(A) . \det(B)$$

but in here since $\det(A) = 0$, we can follow a less complicated method to prove that. Assume that the product of matrices AB has full rank and there is a matrix D where:

$$C = AB$$

$$CD = I \rightarrow ABD = I$$

since all the matrices A, B, C and D are n by n matrices, we can name the product BD as E so:

$$AE = I \rightarrow E \text{ is an inverse of } A \text{ so } A \text{ is invertable and has full rank}$$

which is obviously not true, so AB is not invertible and does not have full rank.

c)

Assuming $A_{n\times m}$, $x_{n\times 1}$ and $b_{m\times 1}$, we have:

$$Ax = b$$

taking transpose from both sides:

$$(Ax)^T = b^T \rightarrow x^T A^T = b^T$$

Question 3:

We first try to solve the problem using the elementary row operations. We can form the augmented matrix as follows:

$$\left[\begin{array}{ccc|c} \beta & \alpha & \alpha & \gamma \\ \alpha & \beta & \alpha & \gamma \\ \alpha & \alpha & \beta & \gamma \end{array}\right]$$

Subtracting the first row from the second and third one, we have:

$$\left[\begin{array}{ccc|c} \beta & \alpha & \alpha & \gamma \\ \alpha - \beta & \beta - \alpha & 0 & 0 \\ 0 & \alpha - \beta & \beta - \alpha & 0 \end{array}\right]$$

Now if $\alpha = \beta$, all the coefficients of the second and third equation will be zero and the only identifying equation will be:

$$\beta x_1 + \alpha x_2 + \alpha x_3 = \gamma$$

Obviously. there is no unique answer to this equation since the number of the variables is larger than the one.

But if $\alpha \neq \beta$, we have the following system of equations:

$$\beta x_1 + \alpha x_2 + \alpha x_3 = \gamma$$

$$(\alpha - \beta)x_1 + (\beta - \alpha)x_2 = 0$$

$$(\alpha - \beta)x_2 + (\beta - \alpha)x_3 = 0$$

Using equations 2 and 3, it is easy to realize that:

$$x_1 = x_2 = x_3$$

Putting it into the first equation, we have:

$$(\beta + 2\alpha)x = \gamma \rightarrow x = x_1 = x_2 = x_3 = \frac{\gamma}{\beta + 2\alpha}$$

Which is a unique solution for the system of equations.

We can also use the Cramer's rule to derive the variables:

$$x_1 = \frac{\det\left(\begin{bmatrix} \gamma & \alpha & \alpha \\ \gamma & \beta & \alpha \\ \gamma & \alpha & \beta \end{bmatrix}\right)}{\det\left(\begin{bmatrix} \beta & \alpha & \alpha \\ \alpha & \beta & \alpha \\ \alpha & \alpha & \beta \end{bmatrix}\right)} = \frac{\gamma\big((\beta^2 - \alpha^2) - 2\alpha(\beta - \alpha)\big)}{\beta(\beta^2 - \alpha^2) + (-2\alpha^2)(\beta - \alpha)} =$$

$$= \frac{\gamma((\beta - \alpha)(\alpha + \beta - 2\alpha))}{(\beta - \alpha)(\beta^2 + \beta\alpha - 2\alpha^2)} = \frac{\gamma(\beta - \alpha)^2}{(\beta - \alpha)^2(\beta + 2\alpha)} = \frac{\gamma}{\beta + 2\alpha}$$

Using the exact same way, we will get the same results for x_2 , x_3.

Question 4:

a)

one way to show that a matrix has a full rank or not is to calculate the determinant of that matrix. If the determinant is equal to 0 then the matrix does not have full rank. Otherwise, the rows and columns of the matrix are linearly independent and it has full rank. So in here:

$$M = \begin{bmatrix} 0.3 & 0.5 & 0.4 \\ 0.4 & 0.1 & 0.5 \\ 0.3 & 0.4 & 0.1 \end{bmatrix}$$

And the determinant of the matrix M is:

$$\det(M) = 0.3 * \det\begin{bmatrix} 0.1 & 0.5 \\ 0.4 & 0.1 \end{bmatrix} - 0.5 * \det\begin{bmatrix} 0.4 & 0.5 \\ 0.3 & 0.1 \end{bmatrix} + 0.4 * \det\begin{bmatrix} 0.4 & 0.1 \\ 0.3 & 0.4 \end{bmatrix}$$

$$\Rightarrow \det(M) = 0.3(0.01 - 0.2) - 0.5(0.04 - 0.15) + 0.4(0.16 - 0.03) = 0.05 \neq 0$$

So the matrix M has full rank. Now for the second matrix N, which is:

$$N = \begin{bmatrix} 0.7 & -0.5 & -0.4 \\ -0.4 & 0.9 & -0.5 \\ -0.3 & -0.4 & 0.9 \end{bmatrix}$$

Although it is possible to calculate the determinant to realize whether the matrix has full rank or not, We can already say that the matrix N does not have full rank, because the rows of this matrix are not linearly independent and:

$$First\ row + Second\ row = -Third\ row$$

b)

$$\begin{bmatrix} 0.7 & -0.5 & -0.4 \\ -0.4 & 0.9 & -0.5 \\ -0.3 & -0.4 & 0.9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the determinant of the matrix of the coefficients, A, is zero it is possible to obtain a nonzero response for the equation above.

using the second equation, we can write x_3 as a linear function of x_1 , x_2:

$$x_3 = -0.8x_1 + 1.8x_2$$

using equation 1 we have:

$$x_1 \approx 1.19x_2$$

any nonzero vector of x, which satisfies the above conditions is a valid response.

c)

Assume that we have a stochastic matrix A

$$A = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix}$$

So for every column of the matrix we have:

$$\sum_{i=1}^{n} x_{ij} = 1$$

Now we define a matrix B, which is equal to $A - I$.

$$B = \begin{bmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & \ddots & \vdots \\ y_{n1} & \cdots & y_{nn} \end{bmatrix}$$

Now because B=A-I we can easily conclude that for every column of the matrix B:

$$\sum_{i=1}^{n} y_{ij} = 0$$

And this means that the rows of the matrix B are not linearly independent from each other and it is easy to see that for any k:

$$[x_{k1} \quad \cdots \quad x_{kn}] = -\sum_{i \neq k} [x_{i1} \quad \cdots \quad x_{in}]$$

So the matrix B does not have full rank.

We can get the exact same results for the matrix $C = I - A$.

Question 5:

a) We know that the sum of the column elements is

$$\sum_{i=1}^{30} a_{ij} = 1$$

This is according to the assumptions of the question. The sum of the endorsements given by any student should be equal to one.

b) Now we set

$$B_i = \sum_{j=1}^{30} a_{ij}$$

This is equal to the total amount of endorsement that the individual i get from her classmates and it is a good measure for determining the popularity of each student.

c) It is easy to see that the matrix A is a stochastic matrix, and the sum of the elements of each column is equal to one (Note that we assumed here that all of the students should give endorsements about their classmates). So as we have seen before if A is a stochastic matrix then $A - I$ does not have full ranks.

d) Consider the case where we divide the class of 30 students into 15 groups and each group consists of only two students and these two only endorse each other and no one else. For example we can assume that student 1 only endorses student 2 and vice versa, and we have the same situation for students 3 and 4 till we get to students 29 and 30. Reminding the main equation:

$$Ax = x$$

We immediately realize that with the above conditions we will get the following results:

$$x_1 = x_2, x_3 = x_4, \dots \quad , x_{29} = x_{30}$$

Moreover, we have the condition that $\sum_{i=1}^{30} x_i = \frac{1}{30}$.

Consequently, we have 30 variables but we only have 16 equations to solve them, and this means that we have infinite solutions for these equations.