Mathematics for Economists
ECON-C1000
Spring 2022
Juuso Välimäki
juuso.valimaki@aalto.fi

Problem Set 2: Due May 5, 2022

1. Chain rule and implicit functions:

   (a) Let $f(x,y) = 4xy, x(t) = 5t^3 + 1$ and $y(t) = t^2 + 5t$. Compute the derivative of $f(x(t), y(t))$ with respect to t first by chain rule and then by plugging in the formulas for $x(t), y(t)$ and taking the derivative. Compare your solutions (and the ease of getting at the solutions).

   (b) Find the points where the gradient of the following functions is zero. We'll see next week how to decide if these points are minima or maxima.

      i. $f(x,y) = 8x^2 + 8xy - 3x^2 + 4y^2 + 1$

      ii. $f(x,y) = x + 2e^y - e^x - e^{2y}$

   (c) Consider the equation:

   $$f(y,x,z) = 2y^3x^2 + z^3 + 2yxz - 5 = 0.$$

   Treat y as the endogenous variable. Can you use implicit function theorem around $(y,x,z) = (1,1,1)$? Compute $\frac{dy}{dz}$ and $\frac{dy}{dz}$ around this point.

2. Consider the production function $Y(K,L) = A\big(\alpha K^\rho + (1-\alpha)L^\rho\big)^{\frac{1}{\rho}}$ called constant elasticity of substitution function or CES-function.

   (a) Calculate and interpret the partial derivatives of this function with respect to capital $K > 0$ and labor $L > 0$.

   (b) Show that

   $$\lim_{\rho \to 0} Y(K,L) = AK^\alpha L^{1-\alpha}.$$

   In words, show that the CES-function converges to the Cobb-Douglas -function. (Hint: consider $lnY(K,L)$ and use l'Hopital's rule as $\rho \to 0$.)

3. Consider the system of equations:

$$\frac{\alpha}{y_1} - y_3 z_1 = 0$$
$$\frac{\beta}{y_2} - y_3 z_2 = 0$$
$$z_1 y_1 + z_2 y_2 - z_3 = 0$$

(a) Show that the system is satisfied at point

$$(y_1, y_2, y_3, z_1, z_2, z_3) = (1, 1, 1, \alpha, \beta, \alpha + \beta)$$

for $\alpha, \beta > 0$

(b) Show that you can take (y_1, y_2, y_3) as endogenous variables and use the implicit function theorem there.

4. Consider the following pair of functions for $x_1 \geq 0, x_2 \geq 0$, $(x_1, x_2) \neq (0, 0)$:

$$f_1(x_1, x_2; c_1, c_2) = \frac{x_1}{x_1 + x_2} - c_1 x_1,$$

$$f_2(x_1, x_2; c_1, c_2) = \frac{x_2}{x_1 + x_2} - c_2 x_2,$$

where $c_1, c_2 > 0$. For $(x_1, x_2) = (0, 0)$, set $f_1(0, 0; c_1, c_2) = f_2(0, 0; c_1, c_2) = 0$.

(a) Compute for each $i \in \{1, 2\}$, $\frac{\partial f_i(x_1, x_2; c_1, c_2)}{\partial x_i}$ for $x_1, x_2 > 0$. How do these partial derivatives depend on $\{x_1, x_2\}$?

(b) Form the system of equations where for each $i \in \{1, 2\}$, $\frac{\partial f_i(x_1, x_2; c_1, c_2)}{\partial x_i} = 0$. Solve the system.

(c) (Extra Credit) Consider the modified pair of functions:

$$g_1(x_1, x_2; c_1, c_2, r) = \frac{x_1^r}{x_1^r + x_2^r} - c_1 x_1,$$

$$g_2(x_1, x_2; c_1, c_2, r) = \frac{x_2^r}{x_1^r + x_2^r} - c_2 x_2,$$

where $c_1, c_2 > 0, 0 < r < 1$. Form again the system where for each $i \in \{1, 2\}$, $\frac{\partial g_i(x_1, x_2; c_1, c_2, r)}{\partial x_i} = 0$. Show that for $c_1 = c_2 = c$, a symmetric solution $x_1(c, c, r) = x_2(c, c, r) < 1/2$, exists for $0 < r < 1$ but not if r is very large and determine the effect of r on $x_i(c, c, r)$ when the symmetric solution exists.

(d) (Extra Credit) Can you show the existence of a solution in the asymmetric case with $c_1 < c_2$ for small enough r. Can you determine if x_1 is bigger or smaller than x_2 when a solution exists?

(e) (For your information, no question here.) The story behind the problem is the following. Two countries $i \in \{1, 2\}$ are in conflict over a resource of size 1 and prepare for war over it. The outcome in the war depends on the sizes of the armies so that the probability of winning is proportional to the size of the armies: $\Pr\{\text{ i wins }\} = \frac{x_i}{x_1 + x_2}$. The cost of raising an army of size x_i in country i is $c_i x_i$. Then

$$f_1(x_1, x_2; c_1, c_2) = \Pr\{i \text{ wins with army } x_i\} \times 1 - \text{ cost of } x_1.$$

The parameter r in the modification determines how sensitive the probability of winning is to differences in army size.