Mathematics for Economists

Instructor: Juuso Valimaki

Teacher Assistant: Amin Mohazab

amin.mohazabrahimzadeh@gmail.com

Solutions to the problem set 2:

Question 1

a)

First, we try to obtain the derivative using the chain rule:

$$\frac{d}{dt}f(x,y) = \frac{\partial f}{\partial x}x'(t) + \frac{\partial f}{\partial y}y'(t) =$$
$$\big(4y(t)\big)15t^2 + \big(4x(t)\big)(2t+5) =$$
$$(4t^2 + 20t)(15t^2) + (20t^3 + 4)(2t+5) =$$
$$100t^4 + 400t^3 + 8t + 20$$

Now by plugging x(t) and y(t) to f(x,y) and taking the derivative:

$$f\big(x(t), y(t)\big) = 4(5t^3 + 1)(t^2 + 5t) =$$
$$4(5t^4 + 25t^4 + t^2 + 5t) =$$
$$20t^5 + 100t^4 + 4t^2 + 20t$$

Now by taking a derivative with respect to t, we have:

$$\frac{df}{dt} = 100t^4 + 400t^3 + 8t + 20$$

Which is the same as before.

b)

$$f(x,y) = 8x^3 + 8xy - 3x^2 + 4y^2 + 1$$
$$\frac{df}{dx} = 0 \rightarrow 24x^2 + 8y - 6x = 0$$
$$\frac{df}{dy} = 0 \rightarrow 8x + 8y = 0 \rightarrow x = -y$$

using the last equation we have

$$24x^2 - 14x = 0 \rightarrow 2x(12x - 7) = 0$$

so

$$x = 0\,, y = 0$$
$$x = \frac{7}{12}\,, y = -\frac{7}{12}$$

for the second equation:

$$f(x,y) = x + 2e^y - e^x - e^{2y}$$

$$\frac{df}{dx} = 0 \to 1 - e^x = 0 \to x = 0$$

$$\frac{df}{dy} = 0 \to 2e^y - 2e^{2y} = 0 \to e^y = e^{2y} \to y = 0$$

c)

Since y is the endogenous function, we can write:

$$f(x, y(x,z), z) = y^3(x,z)x^2 + z^3 + xy(x,z)z - 3 = 0$$

To use implicit function theorem, we need two conditions to be satisfied:

- Function f should be continuously differentiable at the point (1,1,1), which is clearly satisfied
- And $\frac{\partial f(\hat{x},\hat{y},\hat{z})}{\partial y} \neq 0$. To see if we have this condition:

$$\frac{\partial f(x,y,z)}{\partial y} = 6y^2x^2 + 2xz = 8$$

So we can use the implicit function theorem here. Taking a derivative from function f with respect to x, we have:

$$f'(x,z) = \frac{\partial f}{\partial x} + y'_x \frac{\partial f}{\partial y} + z'_x \frac{\partial f}{\partial z}$$

But we know that the derivative of z with respect to x is equal to zero, so:

$$\frac{df}{dx}(x,z) = \frac{\partial f}{\partial x} + y'_x \frac{\partial f}{\partial y} = 0 \Rightarrow y'_x = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{4y^3x + 2yz}{6y^2x^2 + 2xz} = -\frac{3}{4}$$

Using an exact same procedure:

$$\frac{df}{dz}(x,z) = y'_z \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = 0 \Rightarrow y'_z = \frac{-\frac{\partial f}{\partial z}}{\frac{\partial f}{\partial y}} = -\frac{3z^2 + 2xy}{6y^2x^2 + 2xz} = -\frac{5}{8}$$

Question 2:

a)

$$Y(K,L) = A(\alpha K^\rho + (1-\alpha)L^\rho)^{\frac{1}{\rho}}$$

$$\frac{d}{dK}Y(K,L) = A\alpha K^{\rho-1}(\alpha K^\rho + (1-\alpha)L^\rho)^{\frac{1}{\rho}-1}$$

$$\frac{d}{dL}Y(K,L) = A(1-\alpha)L^{\rho-1}(\alpha K^\rho + (1-\alpha)L^\rho)^{\frac{1}{\rho}-1}$$

Elasticity of substitution is defined as:

$$\varepsilon = \frac{\Delta \frac{k}{l}}{\Delta MRS}$$

where the denominator is:

$$\Delta MRS = \Delta \frac{dk}{dl} = \Delta \frac{\frac{dY}{dl}}{\frac{dY}{dk}}$$

It can be proved that for the CES function we have:

$$\varepsilon = \frac{1}{1-\rho}$$

b)

$$Y(K,L) = A(\alpha K^\rho + (1-\alpha)L^\rho)^{\frac{1}{\rho}}$$
$$\lim_{\rho \to 0} Y(K,L) = ?$$

To solve this problem we first take logarithms from both sides and then use l'Hopital's rule. So

$$\log Y = \log A + \frac{\log(\alpha K^\rho + (1-\alpha)L^\rho)}{\rho}$$
$$\lim_{\rho \to 0} \log Y(K,L) = \log A + \lim_{\rho \to 0} \frac{\log(\alpha K^\rho + (1-\alpha)L^\rho)}{\rho}$$

using l'Hopital's rule, we have

$$\lim_{\rho \to 0} \log Y(K,L) = \log A + \lim_{\rho \to 0} \frac{(\alpha K^\rho \log K + (1-\alpha)L^\rho \log L)}{\alpha K^\rho + (1-\alpha)L^\rho}$$

$$= \log A + \alpha \log K + (1-\alpha)\log L = \log(AK^\alpha L^{1-\alpha})$$

$$\Rightarrow \lim_{\rho \to 0} Y(K,L) = AK^\alpha L^{1-\alpha}$$

Question 3.

a) We have the system of equations:

$$\frac{\alpha}{y_1} - y_3 z_1 = 0$$
$$\frac{\beta}{y_2} - y_3 z_2 = 0$$
$$z_1 y_1 + z_2 y_2 - z_3 = 0$$

Now at the point:

$$(y_1, y_2, y_3, z_1, z_2, z_3) = (1,1,1, \alpha, \beta, \alpha + \beta)$$

We have:

$$\alpha - \alpha = 0$$
$$\beta - \beta = 0$$
$$\alpha + \beta - (\alpha + \beta) = 0$$

So the system is satisfied at this point.

b) Let's write the system of the equations in the following form:

$$f_1(y_1, y_2, y_3, z_1, z_2, z_3) = \frac{\alpha}{y_1} - y_3 z_1 = 0$$
$$f_2(y_1, y_2, y_3, z_1, z_2, z_3) = \frac{\beta}{y_2} - y_3 z_2 = 0$$
$$f_3(y_1, y_2, y_3, z_1, z_2, z_3) = z_1 y_1 + z_2 y_2 - z_3 = 0$$

And we know that the equations are satisfied at the point:

$$(y_1, y_2, y_3, z_1, z_2, z_3) = (1,1,1, \alpha, \beta, \alpha + \beta)$$

So we make the matrices of the partial derivatives:

$$D_y f(\hat{y}, \hat{z}) = \begin{bmatrix} \frac{\partial f_1(\hat{y}, \hat{z})}{\partial y_1} & \frac{\partial f_1(\hat{y}, \hat{z})}{\partial y_2} & \frac{\partial f_1(\hat{y}, \hat{z})}{\partial y_3} \\ \frac{\partial f_2(\hat{y}, \hat{z})}{\partial y_1} & \frac{\partial f_2(\hat{y}, \hat{z})}{\partial y_2} & \frac{\partial f_2(\hat{y}, \hat{z})}{\partial y_3} \\ \frac{\partial f_3(\hat{y}, \hat{z})}{\partial y_1} & \frac{\partial f_3(\hat{y}, \hat{z})}{\partial y_2} & \frac{\partial f_3(\hat{y}, \hat{z})}{\partial y_3} \end{bmatrix} = \begin{bmatrix} -\frac{\alpha}{\hat{y}_1^2} & 0 & -\hat{z}_1 \\ 0 & -\frac{\beta}{\hat{y}_2^2} & -\hat{z}_2 \\ \hat{z}_1 & \hat{z}_2 & 0 \end{bmatrix} = \begin{bmatrix} -\alpha & 0 & -\alpha \\ 0 & -\beta & -\beta \\ \alpha & \beta & 0 \end{bmatrix}$$

$$\det\left(D_y f(\hat{y}, \hat{z})\right) = -\alpha\beta(\alpha + \beta)$$

And since

$$\alpha, \beta > 0$$

$$\det\left(D_y f(\hat{y}, \hat{z})\right) \neq 0$$

So we have the necessary condition to use the implicit function theorem to obtain $\frac{dy}{dz}$.

Now we have:

$$dy = [D_y f(\hat{y}, \hat{z})]^{-1} [D_z f(\hat{y}, \hat{z})] dz$$

Question 4:

a)

$$f_1(x_1, x_2; c_1, c_2) = \frac{x_1}{x_1 + x_2} - c_1 x_1$$

$$f_2(x_1, x_2; c_1, c_2) = \frac{x_2}{x_1 + x_2} - c_2 x_2$$

we first compute the partial derivatives with respect to x_1, x_2:

$$\frac{\partial f_1}{\partial x_1} = \frac{x_1 + x_2 - x_1}{(x_1 + x_2)^2} - c_1 = \frac{x_2}{(x_1 + x_2)^2} - c_1$$

$$\frac{\partial f_2}{\partial x_2} = \frac{x_2 + x_1 - x_2}{(x_1 + x_2)^2} - c_2 = \frac{x_1}{(x_1 + x_2)^2} - c_2$$

b)

the system of equations:

$$\frac{x_2}{(x_1 + x_2)^2} - c_1 = 0$$

$$\frac{x_1}{(x_1 + x_2)^2} - c_2 = 0$$

we can easily write the above equations as follows:

$$(x_1 + x_2)^2 = \frac{x_2}{c_1}$$

$$(x_1 + x_2)^2 = \frac{x_1}{c_2}$$

so

$$\frac{x_2}{c_1} = \frac{x_1}{c_2} \rightarrow x_2 = \frac{x_1 c_1}{c_2}$$

putting it into the second equation ($\frac{\partial f_2}{\partial x_2} = 0$), we will have:

$$\frac{x_1}{(x_1 + \frac{x_1 c_1}{c_2})^2} = c_2 \rightarrow \frac{1}{x_1 (1 + \frac{c_1}{c_2})^2} = c_2$$

$$x_1 = \frac{1}{c_2 (\frac{c_1 + c_2}{c_2})^2} \rightarrow x_1 = \frac{c_2}{(c_1 + c_2)^2}$$

and

$$x_2 = \frac{c_1}{(c_1 + c_2)^2}$$

c)

$$g_1 = \frac{x_1^r}{x_1^r + x_2^r} - c x_1$$

$$g_2 = \frac{x_2^r}{x_1^r + x_2^r} - c x_2$$

deriving the partial derivative with respect to x_1, we have:

$$\frac{\partial g_1}{\partial x_1} = \frac{(rx_1^{r-1})(x_1^r + x_2^r) - (rx_1^{r-1})x_1^r}{(x_1^r + x_2^r)^2} = \frac{(rx_1^{r-1})x_2^r}{(x_1^r + x_2^r)^2}$$

so the system of the equations is:

$$\frac{\partial g_1}{\partial x_1} = \frac{(rx_1^{r-1})x_2^r}{(x_1^r + x_2^r)^2} - c = 0$$

$$\frac{\partial g_2}{\partial x_2} = \frac{(rx_2^{r-1})x_1^r}{(x_1^r + x_2^r)^2} - c = 0$$

We start by moving the constants (c) to the other sode of the equations and then divide the first equation to the second one. We then have:

$$x_1^{r-1}x_2^r = x_2^{r-1}x_1^r \rightarrow x_1 = x_2$$

putting it in either of the equations, we get:

$$\frac{rx_1^{2r-1}}{4x_1^{2r}} = c \rightarrow \frac{r}{4x_1} = c \rightarrow x_1 = \frac{r}{4c}$$

and

$$x_1 = x_2 = \frac{r}{4c}$$

d) In the case of asymmetric constants, we have:

$$\frac{(rx_1^{r-1})x_2^r}{(x_1^r + x_2^r)^2} = c_1$$

$$\frac{(rx_2^{r-1})x_1^r}{(x_1^r + x_2^r)^2} = c_2$$

Again, by dividing the first equation to the second one, we have:

$$\frac{x_2}{x_1} = \frac{c_1}{c_2} < 1 \rightarrow x_1 > x_2$$

Now by setting this into the first equation, we have:

$$\frac{rx_1^{2r-1}(\frac{c_1}{c_2})^r}{x_1^{2r}(1 + \frac{c_1}{c_2})^2} = c_1 \rightarrow x_1 = \frac{rc_1^{r-1}}{c_2^{r-2}(c_1 + c_2)^2}$$

and similarly

$$x_2 = \frac{rc_2^{r-1}}{c_1^{r-2}(c_1 + c_2)^2}$$

e)

We know that the payoff of going to the war for each country is:

$$\pi(x_1, x_2, c_1, c_2) = \frac{x_i}{x_1 + x_2} - cx_i$$

since $x_1 = x_2$, the first part is equal to $\frac{1}{2}$ and obviously the second part which is the cost of going to the war shouldn't be greater than $\frac{1}{2}$,otherwise they have no intentions to do that.

$$\pi = \frac{1}{2} - cx$$

Now consider the case where $r > 2$. consequently $cx = \frac{r}{4} > \frac{1}{2}$ and the payoff of going to the war (with any amount of army size) will be negative for the countries.

It will be really useful to plot the two cases of the profit function for firm 1. Assume that firm 2 behave according to the equilibrium $x_2 = \frac{r}{4c}$ and c=0.005 and we plot the profit function for two values of r, r=0.5 and r=4.

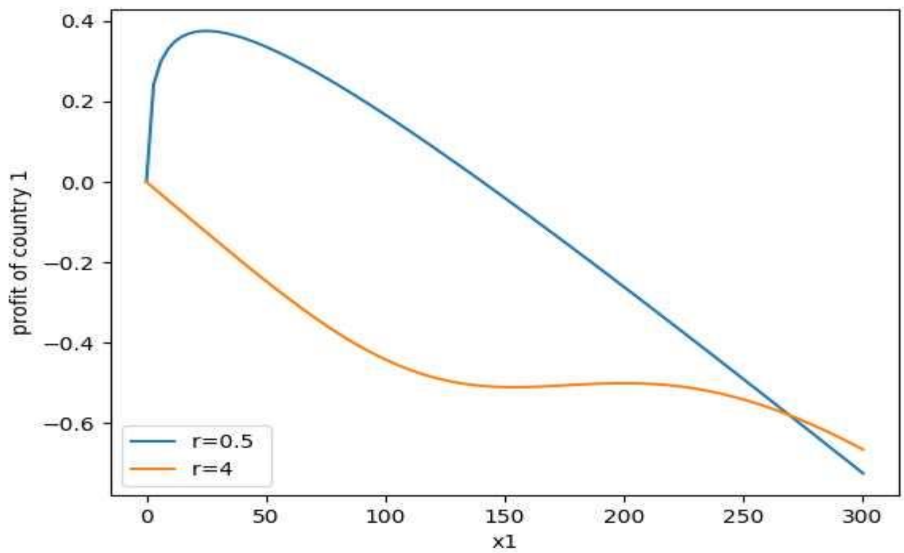

Figure 1

As you see in Figure 1, for $0 < r < 1$ the function is convex and it is easy to obtain the x_1^*. In the second case where $r > 2$, the function f is decreasing at first and it has a local maxima which brings us the negative profit.