Mathematics for Economists

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Solutions to the problem set 3:

Question 1:

a)
i)

$$A = \begin{bmatrix} 1 & 4 \\ 4 & 5 \end{bmatrix}$$

$$M_1 = \det(1) = 1$$

$$M_2 = \det(A) = 5 - 16 = -11$$

For a matrix to be positive definite we should have $M_i > 0$ for all i

and to be negative definite $M_i(-1)^i > 0$

So A is indefinite.

ii)

$$B = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 1 & 4 \\ 3 & 4 & -1 \end{bmatrix}$$

$$M_1 = \det(1) = 1$$

$$M_2 = \det\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right) = 0$$

$$M_3 = \det(B) = 1.\det\left(\begin{bmatrix} 1 & 4 \\ 4 & -1 \end{bmatrix}\right) - 1.\det\left(\begin{bmatrix} -1 & 3 \\ 4 & -1 \end{bmatrix}\right) + 3.\det\left(\begin{bmatrix} -1 & 3 \\ 1 & 4 \end{bmatrix}\right)$$

$$= -17 + 11 - 21 = -27$$

So B is indefinite

iii)

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 9 \end{bmatrix}$$

$$M_1 = \det(1) = 1$$

$$M_2 = \det\left(\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}\right) = 1$$

$$M_3 = \det(B) = 1.\det\left(\begin{bmatrix} 5 & 7 \\ 7 & 9 \end{bmatrix}\right) - 2.\det\left(\begin{bmatrix} 2 & 3 \\ 7 & 9 \end{bmatrix}\right) + 3.\det\left(\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}\right)$$

$$= -4 + 6 - 3 = -1$$

So C is indefinite.

b)

According to the definition, matrix A is positive semi-definite if for any x, $x^t Ax \geq 0$.

Now assuming that $A = X^T X$ where $X_{k\times n}$ and z is any arbitrary vector with length n, we have:

$$z^T(X^T X)z = (Xz)^T(Xz) = \|Xz\|_2^2 \geq 0$$

Note that Xz is a vector of $k \times 1$.

so $X^T X$ is a positive semi-definite matrix .

If we add a constraint "X has rank k" we can prove that $X^T X$ is POSITIVE definite, because then we know there is no non-zero z for which $Xz = 0$.

c)

$$f(x,y) = x^2 + 2bxy + 4y^2 - 3x + 2y + 7$$

to find the critical point of the function f, we should set the gradient equal to zero, so:

$$\frac{df}{dx} = 2x + 2by = 3$$

$$\frac{df}{dy} = 2bx + 8y = -2$$

Multiplying the first equation by b and subtracting the second one from it, we have:

$$y = \frac{3b+2}{2(b^2-4)}$$

Using the same procedure for x

$$x = \frac{b+6}{4-b^2}$$

so $b \neq 2, -2$.

We then form the Hessian matrix of the coefficients:

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 2b \\ 2b & 8 \end{bmatrix}$$

Since the critical point is a global minimum, the Hessian matrix should be positive definite, so

$$\det(H) = 16 - 4b^2 > 0 \rightarrow -2 < b < 2$$

d)

$$f = x.\boldsymbol{A}x + \boldsymbol{b}x + c$$

$$\text{where } \boldsymbol{A} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \boldsymbol{b} = [d, e]$$

Forming the partial derivatives, we have:

$$\frac{\partial f}{\partial x} = 2ax + 2by + d = 0$$

$$\frac{\partial f}{\partial y} = 2bx + 2cy + e = 0$$

Now what happens if $a = b = c \neq 0$ but $d \neq e$. The coefficients are the same but the constants are different. Obviously, it is not possible to solve the system of equations and it has no critical points.

Question 2:

a)

$$f(x, y) = -1 + 3e^{2x}y^2 - 6x - 6y$$

$$\frac{df}{dx} = 6e^{2x}y^2 - 6$$

$$\frac{df}{dy} = 6e^{2x}y - 6$$

$$Hf = \begin{bmatrix} \frac{d^2f}{dxdx} & \frac{d^2f}{dxdy} \\ \frac{d^2f}{dydx} & \frac{d^2f}{dydy} \end{bmatrix} = \begin{bmatrix} 12e^{2x}y^2 & 12e^{2x}y \\ 12e^{2x}y & 6e^{2x} \end{bmatrix} = \begin{bmatrix} 12 & 12 \\ 12 & 6 \end{bmatrix}$$

$$\det(Hf) = -72$$

At point (0,1) partial derivatives are zero and the determinant of the hessian matrix is negative, so we have 2 eigen values with different signs and it is a **saddle point**.

b)

$$f(x, y, z) = x^{\frac{1}{3}}y^{\frac{1}{2}}z - \frac{2}{3}x - y - \frac{1}{3}z$$

$$\frac{df}{dx} = \frac{1}{3}x^{-\frac{2}{3}}y^{\frac{1}{2}}z - \frac{2}{3}$$

$$\frac{df}{dy} = \frac{1}{2}x^{\frac{1}{3}}y^{-\frac{1}{2}}z - 1$$

$$\frac{\partial f}{\partial z} = x^{\frac{1}{3}}y^{\frac{1}{2}} - \frac{1}{3}$$

At point (1,1,2) we have $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ but $\frac{\partial f}{\partial z} = \frac{2}{3} \neq 0$, so it is not a critical point. Though it might be useful to form the Hessian matrix and see its properties:

$$Hf = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial yz} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix} = \begin{bmatrix} -\frac{2}{9} x^{-\frac{5}{3}} y^{\frac{1}{2}} z & \frac{1}{6} x^{-\frac{2}{3}} y^{-\frac{1}{2}} z & \frac{1}{3} x^{-\frac{2}{3}} y^{-\frac{1}{2}} \\ \frac{1}{6} x^{-\frac{2}{3}} y^{-\frac{1}{2}} z & -\frac{1}{4} x^{\frac{1}{3}} y^{-\frac{3}{2}} z & \frac{1}{2} x^{\frac{1}{3}} y^{-\frac{1}{2}} \\ \frac{1}{3} x^{-\frac{2}{3}} y^{-\frac{1}{2}} & \frac{1}{2} x^{\frac{1}{3}} y^{-\frac{1}{2}} & 0 \end{bmatrix}$$

$$Hf = \begin{bmatrix} -\frac{4}{9} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 \end{bmatrix}$$

so

$$M_1 = -\frac{4}{9}$$

$$M_2 = \det\left(\begin{bmatrix} -\frac{4}{9} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{2} \end{bmatrix}\right) = \frac{1}{9}$$

$$M_3 = -\frac{1}{2}\det\left(\begin{bmatrix} -\frac{4}{9} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} \end{bmatrix}\right) + \frac{1}{3}\det\left(\begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\right) = \frac{5}{18}$$

so the matrix Hf is indefinite.

Question 3:

$$f(x) = \frac{x}{(x-1)^2 + 1} = \frac{x}{x^2 - 2x + 2}$$

$$f'(x) = \frac{x^2 - 2x + 2 - x(2x-2)}{(x^2 - 2x + 2)^2} = \frac{-x^2 + 2}{(x^2 - 2x + 2)^2}$$

$$f'(x) = 0 \rightarrow x = \sqrt{2}, -\sqrt{2}$$

To derive the second order taylor approximation, we also need f'', so

$$f''(x) = \frac{(-2x)(x^2 - 2x + 2)^2 - (-x^2 + 2)(2(2x-2)(x^2 - 2x + 2))}{(x^2 - 2x + 2)^4}$$

The second term in the numerator is equal to zero at $x = \sqrt{2}, -\sqrt{2}$, so

$$f''(\sqrt{2}) = \frac{-2\sqrt{2}}{(4 - 2\sqrt{2})^2} < 0 \rightarrow local\ maxima$$

and

$$f''(-\sqrt{2}) = \frac{2\sqrt{2}}{(4+2\sqrt{2})^2} > 0 \rightarrow local\ minima$$

Taylor approximation around $x = \sqrt{2}$

$$f(\sqrt{2}+h) = f(\sqrt{2}) + f'(\sqrt{2})h + \frac{1}{2}f''(\sqrt{2})h^2 = \frac{\sqrt{2}}{(\sqrt{2}-1)^2+1} - \frac{\sqrt{2}}{(4-2\sqrt{2})^2}h^2$$

Taylor approximation around $x = -\sqrt{2}$

$$f(-\sqrt{2}+h) = f(-\sqrt{2}) + f'(-\sqrt{2})h + \frac{1}{2}f''(-\sqrt{2})h^2 = \frac{-\sqrt{2}}{(-\sqrt{2}-1)^2+1} + \frac{\sqrt{2}}{(4+2\sqrt{2})^2}h^2$$

Question 4:

a)

R is the overall return

$$R = \sum x_i R_i = x_A R_A + x_B R_B$$

$$E(R) = E(x_A R_A + x_B R_B) = x_A E(R_A) + x_B E(R_B)$$

$$\mu = x_A \mu_A + x_B \mu_B$$

σ is the overall variance

$$\sigma = E[(R-\mu)^2] =$$

$$E[(x_A R_A + x_B R_B - x_A \mu_A - x_B \mu_B)^2] =$$

$$E\left[\left(x_A(R_A-\mu_A) + x_B(R_A-\mu_A)\right)^2\right] =$$

$${x_A}^2 E[(R_A-\mu_A)^2] + {x_B}^2 E[(R_B-\mu_B)^2] + 2x_A x_B E[(R_A-\mu_A)(R_B-\mu_B)]$$

$$\sigma = x_A^2 \sigma_A + x_B^2 \sigma_B + 2x_A x_B \sigma_{AB}$$

b)

Our objective function is:

$$\min_{x_A, x_B} x_A^2 \sigma_A + x_B^2 \sigma_B + 2x_A x_B \sigma_{AB}$$

$$st. x_A + x_B = x\ , \sigma_{AB} = 0$$

so equivalently

$$\min_{x_A, x_B} x_A^2 \sigma_A + x_B^2 \sigma_B$$

$$st. x_A + x_B = x$$

According to the condition

$$x_B = x - x_A$$

$$\min_{x_A} x_A^2\sigma_A + (x - x_A)^2\sigma_B$$

Using the first order condition:

$$\frac{d\sigma}{dx_A} = 2x_A\sigma_A - 2(x - x_A)\sigma_B = 0$$

$$x_A = \frac{\sigma_B x}{\sigma_A + \sigma_B} \ and \ x_B = \frac{\sigma_A x}{\sigma_A + \sigma_B}$$

c)

The utility of the investor is $u = \gamma\mu - \sigma$, so he tries to solve the following optimization function:

$$\max_{x_A, x_B} \gamma(x_A\mu_A + x_B\mu_B) - x_A^2\sigma_A - x_B^2\sigma_B - 2x_Ax_B\sigma_{AB}$$

We first form the first order condition by taking partial derivatives with respect to the portfolio variables:

$$\frac{\partial u}{\partial x_A} = \gamma\mu_A - 2x_A\sigma_A - 2x_B\sigma_{AB} = 0$$

$$\frac{\partial u}{\partial x_B} = \gamma\mu_B - 2x_B\sigma_B - 2x_A\sigma_{AB} = 0$$

so the system of equations is

$$\gamma\mu_A = 2x_A\sigma_A + 2x_B\sigma_{AB}$$

$$\gamma\mu_B = 2x_B\sigma_B + 2x_A\sigma_{AB}$$

Using the second equation, we can write x_B as a function of x_A:

$$x_B = \frac{\gamma\mu_B - 2x_A\sigma_{AB}}{2\sigma_B}$$

We then put it inside the first equation:

$$\gamma\mu_A = 2x_A\sigma_A + 2\sigma_{AB}\left(\frac{\gamma\mu_B - 2x_A\sigma_{AB}}{2\sigma_B}\right)$$

$$2x_A(\sigma_A\sigma_B - \sigma_{AB}^2) = \gamma(\mu_A\sigma_B - \mu_B\sigma_{AB})$$

$$x_A = \frac{\gamma(\mu_A\sigma_B - \mu_B\sigma_{AB})}{2(\sigma_A\sigma_B - \sigma_{AB}^2)}$$

and similarly we have:

$$x_B = \frac{\gamma(\mu_B\sigma_A - \mu_A\sigma_{AB})}{2(\sigma_A\sigma_B - \sigma_{AB}^2)}$$

We can also check for the Hessian matrix of the equation:

$$H = \begin{bmatrix} -2\sigma_A & -2\sigma_{AB} \\ -2\sigma_{AB} & -2\sigma_B \end{bmatrix}$$

where

$$M_1 = -2\sigma_A < 0$$

$$M_2 = \sigma_A\sigma_B - \sigma_{AB}^2$$

If $M_2 > 0$, the matrix H is negative definite and the critical point is a global maximum.

Question 5:

a)

composite function $v(x, y) = f(u(x, y))$

$$MRS_{u(x,y)} = \frac{\frac{\partial u(x_0, y_0)}{\partial x}}{\frac{\partial u(x_0, y_0)}{\partial y}}$$

a,b)

$$MRS_{v(x,y)} = \frac{\frac{\partial v(x_0, y_0)}{\partial x}}{\frac{\partial v(x_0, y_0)}{\partial y}} = \frac{\frac{\partial f(u(x_0, y_0))}{\partial u} \cdot \frac{\partial u(x_0, y_0)}{\partial x}}{\frac{\partial f(u(x_0, y_0))}{\partial u} \cdot \frac{\partial u(x_0, y_0)}{\partial y}} = \frac{\frac{\partial u(x_0, y_0)}{\partial x}}{\frac{\partial u(x_0, y_0)}{\partial y}} = MRS_{u(x,y)}$$

c)

Definition of the homogenous function:

A real valued function $f(x_1, x_2, \dots, x_n)$ is homogenous of degree k if for all $t > 0$:

$$f(tx_1, tx_2, \dots, tx_n) = t^k f(x_1, x_2, \dots, x_n)$$

so considering u as a homogenous function of degree k, we have:

$$MRS_{u(tx,ty)} = \frac{\frac{\partial u(tx_0, ty_0)}{\partial x}}{\frac{\partial u(tx_0, ty_0)}{\partial y}} = \frac{\frac{t^k \partial u(x_0, y_0)}{\partial x}}{\frac{t^k \partial u(x_0, y_0)}{\partial y}} = \frac{\frac{\partial u(x_0, y_0)}{\partial x}}{\frac{\partial u(x_0, y_0)}{\partial y}} = MRS_{u(x,y)}$$

Question 6:

a)

We assume that prices are given from the outside of the firm, so p, w and r are exogenous variables. On the other hand, it is possible for us to calculate the exact amount of the capital and labour to increase out profit so l and k are endogenous variables.

b)

our endogenous variables are k and l, so:

$$\frac{\partial f(k,l)}{\partial k} - \frac{r}{p} = 0$$

$$\frac{\partial f(k,l)}{\partial l} - \frac{w}{p} = 0$$

c)

We rewrite the problem as follows:

$$f_1(k,l;p,r,w) = g(k,l) - \frac{r}{p}$$
$$f_2(k,l;p,r,w) = h(k,l) - \frac{w}{p}$$

where $g(k,l) = \frac{\partial f(k,l)}{\partial k}$ and $h(k,l) = \frac{\partial f(k,l)}{\partial l}$. We should assume that the system of the equations are satisfied at $(\bar{k}, \bar{l}, \bar{p}, \bar{r}, \bar{w})$, so in the next step we make the matrices of partial derivative (we assume y as the endogenous and x as the exogenous variables):

$$D_y f(\bar{k}, \bar{l}; \bar{p}, \bar{r}, \bar{w}) = \begin{bmatrix} \frac{\partial f_1}{\partial k} & \frac{\partial f_1}{\partial l} \\ \frac{\partial f_2}{\partial k} & \frac{\partial f_2}{\partial l} \end{bmatrix} = \begin{bmatrix} \frac{\partial g(\bar{k},\bar{l})}{\partial k} & \frac{\partial g(\bar{k},\bar{l})}{\partial l} \\ \frac{\partial h(\bar{k},\bar{l})}{\partial k} & \frac{\partial h(\bar{k},\bar{l})}{\partial l} \end{bmatrix}$$

And

$$D_x f(\bar{k}, \bar{l}; \bar{p}, \bar{r}, \bar{w}) = \begin{bmatrix} \frac{\partial f_1}{\partial p} & \frac{\partial f_1}{\partial r} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial p} & \frac{\partial f_2}{\partial r} & \frac{\partial f_2}{\partial w} \end{bmatrix} = \begin{bmatrix} \frac{r}{p^2} & -\frac{1}{p} & 0 \\ \frac{w}{p^2} & 0 & -\frac{1}{p} \end{bmatrix}$$

Now assuming the fact that functions f and g are continuously differentiable at $(\bar{k}, \bar{l}, \bar{p}, \bar{r}, \bar{w})$, the necessary condition is:

$$\det\,(D_y f(\bar{k}, \bar{l}; \bar{p}, \bar{r}, \bar{w})) \neq 0$$

so:

$$\frac{\partial g(\bar{k},\bar{l})}{\partial k} . \frac{\partial h(\bar{k},\bar{l})}{\partial l} \neq \frac{\partial g(\bar{k},\bar{l})}{\partial l} \frac{\partial h(\bar{k},\bar{l})}{\partial k}$$

d)

Using implicit function theorem we have:

$$D_y f(\hat{y}, \hat{x}) dy + D_x f(\hat{y}, \hat{x}) dx = 0$$

So

$$\begin{bmatrix} \frac{\partial g(\bar{k},\bar{l})}{\partial k} & \frac{\partial g(\bar{k},\bar{l})}{\partial l} \\ \frac{\partial h(\bar{k},\bar{l})}{\partial k} & \frac{\partial h(\bar{k},\bar{l})}{\partial l} \end{bmatrix} \begin{bmatrix} dk \\ dl \end{bmatrix} + \begin{bmatrix} \frac{r}{p^2} & -\frac{1}{p} & 0 \\ \frac{w}{p^2} & 0 & -\frac{1}{p} \end{bmatrix} \begin{bmatrix} dp \\ dr \\ dw \end{bmatrix} = 0$$

Using Cramer's rule:

$$dk = \frac{det \begin{bmatrix} \frac{1}{\rho} & \frac{\partial g(\bar{k},\bar{l})}{\partial l} \\ 0 & \frac{\partial h(\bar{k},\bar{l})}{\partial l} \end{bmatrix} dr}{\det \begin{bmatrix} \frac{\partial g(\bar{k},\bar{l})}{\partial k} & \frac{\partial g(\bar{k},\bar{l})}{\partial l} \\ \frac{\partial h(\bar{k},\bar{l})}{\partial k} & \frac{\partial h(\bar{k},\bar{l})}{\partial l} \end{bmatrix}}$$

We assume that the denominator of the above formula is always positive. so

$$\frac{\partial g(\bar{k},\bar{l})}{\partial k} \cdot \frac{\partial h(\bar{k},\bar{l})}{\partial l} > \frac{\partial g(\bar{k},\bar{l})}{\partial l} \frac{\partial h(\bar{k},\bar{l})}{\partial k}$$

why? It simply means that the cross effects of k and l are not too strong over each other. In other words, adding Labor does not change the marginal product of k (MP_k) too much and vice versa.

considering the previous assumption, we easily conclude that the sign of $\frac{dk}{dr}$ is the same as the sign of

$$\frac{1}{\rho} \frac{\partial h(\bar{k},\bar{l})}{\partial l}$$

We can compute $\frac{dl}{dr}$ by doing the same:

$$dl = \frac{det \begin{bmatrix} \frac{\partial g(\bar{k},\bar{l})}{\partial k} & \frac{1}{\rho} \\ \frac{\partial h(\bar{k},\bar{l})}{\partial k} & 0 \end{bmatrix} dr}{\det \begin{bmatrix} \frac{\partial g(\bar{k},\bar{l})}{\partial k} & \frac{\partial g(\bar{k},\bar{l})}{\partial l} \\ \frac{\partial h(\bar{k},\bar{l})}{\partial k} & \frac{\partial h(\bar{k},\bar{l})}{\partial l} \end{bmatrix}}$$

And the sign of $\frac{dl}{dr}$ is the same as the sign of

$$-\frac{1}{\rho} \frac{\partial h(\bar{k},\bar{l})}{\partial k}$$