Mathematics for Economists
ECON-C1000
Spring 2022
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Problem Set 3: Due May 12, 2022

1. Consider the following matrices.

   (a) Determine if the matrices are positive or negative definite:

   $$\text{i)} \begin{pmatrix} 1 & 4 \\ 4 & 5 \end{pmatrix} \qquad \text{ii)} \begin{pmatrix} 1 & -1 & 3 \\ -1 & 1 & 4 \\ 3 & 4 & -1 \end{pmatrix} \qquad \text{ii)} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 9 \end{pmatrix}$$

   (b) Show that all matrices of the form $X^\top X$ (where X is an arbitrary $k \times n$ matrix) are positive semi-definite. When are they positive definite?.

   (c) Find the critical points of the following function:

   $$f(x, y) = (x^2 + 2bxy + 4y^2) - 3x + 2y + 7,$$

   and determine the values of b such that the critical points are minima.

   (d) Give an example of a quadratic function $\boldsymbol{x} \cdot \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}\boldsymbol{x} + c$, that has $a_{i,j} \neq 0$ for all i, j and that has no critical points.

2. Determine the nature of the critical points:

   (a) $f(x, y) = -1 + 3e^{2x}y^2 - 6x - 6y$ at $x = 0, y = 1$.

   (b) $f(x, y, z) = x^{\frac{1}{3}}y^{\frac{1}{2}}z - \frac{2}{3}x - y - \frac{1}{3}z$ at $x = 1, y = 1, z = 2$.

3. Consider the function:

   $$f(x) = \frac{x}{(x-1)^2 + 1}.$$

(a) Find any critical points $\hat{x}$ where $f'(\hat{x}) = 0$.

(b) Find the second order Taylor approximation to the function around $\hat{x}$

$$f(\hat{x} + h) = f(\hat{x}) + f'(\hat{x})h + \frac{1}{2}f''(\hat{x})h^2$$

and determine if f has a local minimum or maximum at $\hat{x}$.

4. (Requires concepts from Probability and Statistics) Consider the optimal investment decision between two possible assets A and B. Let R_i denote the return from asset i for $i \in \{A, B\}$. These returns are random variables and denote the expectation of $\mathbb{E}(R_i)$ by μ_i. Recall that the variance of R_i is given by σ_i, where:

$$\sigma_i = \mathbb{E}[(R_i - \mu_i)^2],$$

and the covariance σ_{AB} is given by

$$\sigma_{AB} = \mathbb{E}[(R_A - \mu_A)(R_B - \mu_B)].$$

(a) A portfolio (x_A, x_B) with x_i lists the number of assets of each type. Compute the expected return μ_{x_A,x_B} and the variance σ_{x_A,x_B} of the return for a portfolio as a function of (x_A, x_B).

(b) Assume that the two assets are risky so that $\sigma_A > \sigma_B > 0$ for $i \in \{1, 2\}$, but they are uncorrelated so that $\sigma_{AB} = 0$. If we require that $x_A + x_B = x$ so that the total investment is x, find the portfolio that minimizes the variance.

(c) Suppose that the investor likes expected return but dislikes risk in the form of variance so that her utility function over different portfolios is given by

$$u(x_A, x_B) = \gamma\mu_{x_A,x_B} - \sigma_{x_A,x_B},$$

where $\gamma > 0$ is a parameter measuring the risk tolerance of the investor. Find the first-order condition for the optimal portfolio in the general case where $\sigma_{AB} \neq 0$.

5. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a differentiable utility function, and let $f : \mathbb{R} \to \mathbb{R}$ be differentiable function with a strictly positive derivative for every $x \in \mathbb{R}$. Define the composite function $v(x, y) := f(u(x, y))$. Recall that the Marginal Rate of Substitution of u at a point (x_0, y_0) is

$$MRS_{x,y} = \frac{\frac{\partial u(x_0,y_0)}{\partial x}}{\frac{\partial u(x_0,y_0)}{\partial y}}.$$

   (a) Write the expression of the MRS at (x_0, y_0) for the composite function v.

   (b) Use the chain rule to show that the MRS of u and v at (x_0, y_0) is the same.

   (c) Now assume that u is also homogeneous of degree k. Show that the MRS of u is a homogeneous function of degree zero.

6. (Implicit function theorem without specified functional forms) A competitive firm maximizes its profit by choosing optimally its inputs:

$$\max_{k,l \geq 0} pf(k, l) - wl - rk,$$

   where $f(k, l)$ is the production function, k is capital, l is labor, r is the rental cost of capital, w the market wage for labor and p is the output price.

   (a) What are the natural endogenous variables for this model? What are the exogenous variables?

   (b) Write the first-order conditions for optimal choices of the endogenous variables.

   (c) When can you use the implicit function theorem to determine the changes in the endogenous variables for small changes in the exogenous variables?

   (d) Compute the sign of changes in the endogenous variables when the exogenous variables change.