Mathematics for Economists
ECON-C1000
Spring 2022
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Problem Set 4: Due May 19, 2022

1. Solve the following maximization problem:

$$\max_{x,y} \sqrt{x} + \sqrt{y}$$

subject to

$$\begin{aligned} x + y &\leq 100 \\ x &\leq 40 \\ x, y &\geq 0 \end{aligned}$$

2. A firm chooses its capital input k and labor input l to minimize its production cost $rk + wl$ subject to the constraint that at least $\hat{q}$ units of output must be produced.

   (a) Write down the optimization problem when output q depends on the inputs as follows: $q = k^\alpha l^\beta$, where $1 > \alpha > 0, 1 > \beta > 0$. What is the feasible set for this problem?

   (b) The feasible set is unbounded and therefore not compact. Can you modify the feasible set so that the conditions for Weierstrass theorem hold?

   (c) What are the first-order necessary conditions for this problem? Find the unique point $k(r, w, \hat{q}), l(r, w, \hat{q})$ solving the problem.

   (d) The full optimization problem of the firm also involves choosing the optimal output target $\hat{q}$ at output prices p. Explain in words, how you would solve for this optimum.

3. Prove the following classical mathematical inequality by constrained optimization.

The arithmetic mean $\frac{1}{n}\sum_{i=1}^{n} x_i$ of positive numbers $x_1, ..., x_n$ is always at least as large as their geometric mean $(\prod_{i=1}^{n} x_i)^{\frac{1}{n}}$. To do this solve the following problem for any $y > 0$:

$$\max_{(x_1,...,x_n)\in\mathbb{R}_+^n} (x_1 \cdot x_2 \cdot ...x_n)^{\frac{1}{n}}$$

subject to

$$\sum_{i=1}^{n} x_i = y.$$

(a) Form the Lagrangean. Argue that an optimal solution must exist

(b) Derive the first-order conditions necessary conditions.

(c) Show that there is a single point that satisfies the necessary conditions. Since an optimal point exists, the solution to necessary conditions must be the optimum. Solve for the optimal x_i.

(d) Show that at the optimal choices, the two averages coincide. Otherwise, arithmetic is larger than geometric.

4. Which of the following claims are true and which are false. For false statements, give a counterexample, for true ones, provide a proof.

   (a) If f and g are concave functions on the real line, then $h(x) = f(x)g(x)$ is also concave.

   (b) Finding a point in a ball with radius 2 centered at $(2, 2, 2)$ at the largest distance from origin can be written as maximizing a concave function on a convex set.

   (c) For any two functions f, g defined on the real line and any two points x, y, we have for all $\lambda \in [0, 1]$:

   $$\begin{aligned}&\max\{\lambda f(x) + (1-\lambda)f(y), \lambda g(x) + (1-\lambda)g(y)\}\\ &\leq \lambda \max\{f(x), g(x)\} + (1-\lambda)\max\{f(y), g(y)\}\end{aligned}$$

   Hint: draw a picture.

   (d) If a utility function defined on $\mathbb{R}_+^n$ is both concave and convex, then it can be written as $u(\boldsymbol{x}) = \boldsymbol{b} \cdot \boldsymbol{x} + c$ for some $\boldsymbol{b} \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

5. A sleepy father likes to eat, play with his children and sleep. Let $f \geq 0$ denote the amount of food that the father eats, $c \geq 0$ the amount of time with children and $s \geq 0$ the amount of sleep. His utility function is increasing in all three components and assume that it takes the form:

$$u(f, c, s) = \ln(f + 1) + \ln(c + 1) + \ln(s + 1).$$

   (a) Show that this utility function is a strictly concave function on non-negative vectors in $(f, c, s) \geq (0, 0, 0)$. Therefore any point satisfying the first-order conditions on any convex feasible set is a global maximum.

   (b) Unfortunately to get food, the father must work and working is away from either sleep or time with children. What are the time constraint and the budget constraint for the father if he has 24 hours of total time and the wage (in terms of units of food per hour) is $w > 0$?

   (c) What is the feasible set? Is it a convex set?

   (d) Write the maximization problem, the Lagrangean and the first-order conditions for the problem.

   (e) What is the optimal split between time spent sleeping and playing? Can the non-negativity constraints be binding?

   (f) Solve for the unique point satisfying the necessary conditions. Are the conditions for Weierstrass' theorem satisfied? Do you need to check the definiteness of the bordered Hessian to conclude that you have found a maximum?