**Mathematics for Economists**

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**Problem set 4 Solutions:**

**Question 1:**

$$\max_{x,y} \sqrt{x} + \sqrt{y}$$

$$subject\ to: \quad x + y \leq 100$$

$$x \leq 40$$

$$x, y \geq 0$$

We first form the Lagrangian:

$$L = \sqrt{x} + \sqrt{y} - \lambda_1(x + y - 100) - \lambda_2(x - 40) + \lambda_3 x + \lambda_4 y = 0$$

The first-order conditions:

$$\frac{\partial L}{\partial x} = 0 \Rightarrow \frac{1}{2\sqrt{x}} - \lambda_1 - \lambda_2 + \lambda_3 = 0 \quad (1)$$

$$\frac{\partial L}{\partial y} = 0 \Rightarrow \frac{1}{2\sqrt{y}} - \lambda_1 + \lambda_4 = 0 \quad (2)$$

$$\lambda_1(x + y - 100) = 0 \quad (3)$$

$$\lambda_2(x - 40) = 0 \quad (4)$$

$$\lambda_3 x = 0\,, \lambda_4 y = 0$$

$$x + y \leq 100, \qquad x \leq 40, \qquad x, y \geq 0$$

Since the objective function is an increasing function of x and y, the optimum happens when $x, y > 0$, and as the result $\lambda_3, \lambda_4 = 0$. Putting this into (1) and (2) we have:

$$\frac{1}{2\sqrt{x}} - \lambda_1 - \lambda_2 = 0$$

$$\frac{1}{2\sqrt{y}} - \lambda_1 = 0$$

It is obvious from the second constraint that $\lambda_1 \neq 0$, so constraint (3) binds and

$$x + y = 100$$

To determine the condition of constraint (4), we start with the assumption that $\lambda_2 = 0$, so

$$\frac{1}{2\sqrt{x}} = \lambda_1 = \frac{1}{2\sqrt{y}} \to x = y$$

and using the binding constraint

$$x = y = 50$$

which is not possible since $x \leq 40$, so $\lambda_2 \neq 0$ and constraint (4) also binds

$$x = 40$$

$$x + y = 100 \to y = 60$$

**Question 2:**

a) The optimization problem is as follows:

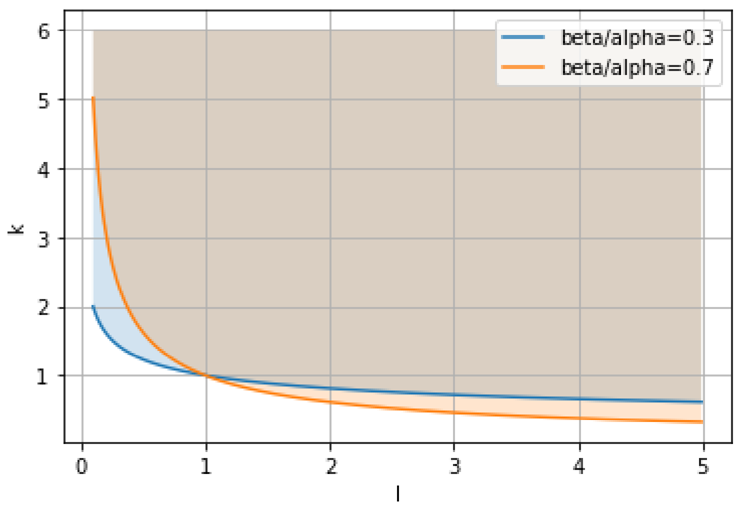
$$\min_{k,l} rk + wl$$

$$st.\ q = k^{\alpha} l^{\beta} \geq \hat{q}$$

The feasible set is defined by the constraints of the problem, so:

$$k^{\alpha} l^{\beta} \geq \hat{q} \to k \geq \frac{\hat{q}^{\frac{1}{\alpha}}}{l^{\frac{\beta}{\alpha}}}$$

Which is an unbounded set over the variables k and l. You can find two examples for $\hat{q} = 1, \frac{\beta}{\alpha} = 0.3, 0.7$ in the following figure:



b)

You can modify the set in any way so that the feasible set becomes closed and compact. We consider the following additional constraint:

$$rk + wl \leq r\bar{k} + w\bar{l}$$

so the optimization problem becomes:

$$\min_{k,l} rk + wl$$

$$st.\ q = k^{\alpha} l^{\beta} \geq \hat{q}$$

$$rk + wl \leq r\bar{k} + w\bar{l}$$

c)

We first form the lagrangian function:

$$L = rk + wl - \lambda_1\left(k^{\alpha} l^{\beta} - \hat{q}\right) + \lambda_2(rk + wl - r\bar{k} - w\bar{l})$$

Now we write the FOC conditions:

$$\frac{\partial L}{\partial k} = r - \alpha\lambda_1 k^{\alpha-1} l^{\beta} + \lambda_2 r = 0$$

$$\frac{\partial L}{\partial l} = w - \beta\lambda_1 k^{\alpha} l^{\beta-1} + \lambda_2 w = 0$$

$$\lambda_1\left(k^{\alpha} l^{\beta} - \hat{q}\right) = 0$$

$$\lambda_2\left(rk + wl - r\bar{k} - w\bar{l}\right) = 0$$

At the first step, we should check for the constraints to be binding or not. Starting from the first constraint, we assume that $\lambda_1 = 0$. From the first FOC condition we have:

$$r + \lambda_2 r = 0 \rightarrow \lambda_2 = -1$$

which is not possible so the first constraint is binding and we have:

$$k^{\alpha} l^{\beta} = \hat{q}$$

To check for the second constraint, we set $\lambda_2 = 0$, so

$$r - \alpha\lambda_1 k^{\alpha-1} l^{\beta} = 0$$

$$w - \beta\lambda_1 k^{\alpha} l^{\beta-1} = 0$$

so

$$\lambda_1 = \frac{r}{\alpha k^{\alpha-1} l^{\beta}} = \frac{w}{\beta k^{\alpha} l^{\beta-1}}$$

$$\rightarrow k = \frac{w\alpha}{r\beta} l$$

Putting it in the equality constraint, we have

$$k^{\alpha}l^{\beta} = \hat{q} \to l^{\alpha+\beta} = \frac{\hat{q}}{(\frac{w\alpha}{r\beta})^{\alpha}} \to l^{*} = \frac{\hat{q}^{\frac{1}{\alpha+\beta}}}{(\frac{w\alpha}{r\beta})^{\frac{\alpha}{\alpha+\beta}}}$$

and

$$k^{*} = \frac{\hat{q}^{\frac{1}{\alpha+\beta}}}{(\frac{w\alpha}{r\beta})^{\frac{-\beta}{\alpha+\beta}}}$$

So the budget constraint is not binding.

d)

The full optimization problem of the firm is as follows:

$$\max_{k,l} pk^{\alpha}l^{\beta} - rk - wl$$

$$st.\ q = k^{\alpha}l^{\beta} \geq \hat{q}$$

$$rk + wl \leq r\bar{k} + w\bar{l}$$

Forming the lagrangian, we have

$$L = pk^{\alpha}l^{\beta} - rk - wl - \lambda_1(k^{\alpha}l^{\beta} - \hat{q}) - \lambda_2(rk + wl - r\bar{k} - w\bar{l})$$

We then form the first order conditions:

$$\frac{\partial L}{\partial k} = p\alpha k^{\alpha-1}l^{\beta} - r - \lambda_1\alpha k^{\alpha-1}l^{\beta} - \lambda_2 r = 0$$

$$\frac{\partial L}{\partial l} = p\beta k^{\alpha}l^{\beta-1} - w - \lambda_1\beta k^{\alpha}l^{\beta-1} - \lambda_2 w = 0$$

$$\lambda_1(k^{\alpha}l^{\beta} - \hat{q}) = 0$$

$$\lambda_2(rk + wl - r\bar{k} - w\bar{l}) = 0$$

**Question 3:**

a)

$$\max_{(x_1,\ldots,x_n)\in R_+^n} (x_1.x_2.\ldots\ x_n)^{\frac{1}{n}}$$

$$\text{st. } \sum_{i=1}^{n} x_i = y$$

We form the Lagrangian first:

$$L(x_1, \ldots, x_n, \lambda) = (x_1.x_2.\ldots\ x_n)^{\frac{1}{n}} + \lambda(\sum_{i=1}^{n} x_i - y)$$

The objective function is obviously continuous. Moreover, all the $x_i$s are positive and sum of them is equal to y, so: $0 \leq x_i \leq y$, so the feasible set is compact and the optimization have solutions. Then we take derivatives of the Lagrangian to get the FOCs.

b)

$$\frac{\partial L}{\partial x_1} = (x_2 \dots x_n)^{\frac{1}{n}}.\frac{1}{n}{x_1}^{\frac{1-n}{n}} - \lambda = 0$$

$$\vdots$$

$$\frac{\partial L}{\partial x_n} = (x_1 \dots x_{n-1})^{\frac{1}{n}}.\frac{1}{n}{x_n}^{\frac{1-n}{n}} - \lambda = 0$$

$$\sum_{i=1}^{n} x_i - y = 0$$

c)

from the first and the second foc:

$$(x_2 \dots x_n)^{\frac{1}{n}}.\frac{1}{n}{x_1}^{\frac{1-n}{n}} = (x_1, x_3 \dots x_n)^{\frac{1}{n}}.\frac{1}{n}{x_2}^{\frac{1-n}{n}}$$

$${x_1}^{\frac{1-n}{n}} x_2^{\frac{1}{n}} = {x_2}^{\frac{1-n}{n}} x_1^{\frac{1}{n}} \Rightarrow x_1 = x_2$$

Using the rest of the constraints we get

$$x_1 = x_2 = \dots = x_n$$

And finally using the last constraint we get

$$x_1^* = x_2^* = \dots = x_n^* = \frac{y}{n}$$

And this is the unique solution of the problem, and in this optimal solution the two averages are equal to each other.

d) At the optimal point:

Arithmetic mean=$\frac{1}{n}\sum_{i=1}^{n} x_i = \frac{y}{n}$

Geometric mean=$(\Pi_{i=1}^{n} x_i)^{\frac{1}{n}} = \frac{y}{n}$

Only in the optimal point, the geometric mean has the maximum value and only in that point it equals to the arithmetic mean. For any other choice of variables x that satisfy the constraint, the geometric mean is less than the maximum value and as the result, it is less than the arithmetic mean.

**Question 4:**

a) There are plenty of functions that we can provide as counterexamples. For example:

$$f(x) = g(x) = x$$

f and g are concave functions (not strictly concave). For the multiplication of them we have

$$h(x) = f(x).h(x) = x^2$$

which is a convex function.

b) The optimization function is as follows:

$$\max_{x,y,z} x^2 + y^2 + z^2$$

$$st.(x-2)^2 + (y-2)^2 + (z-2)^2 \leq 4$$

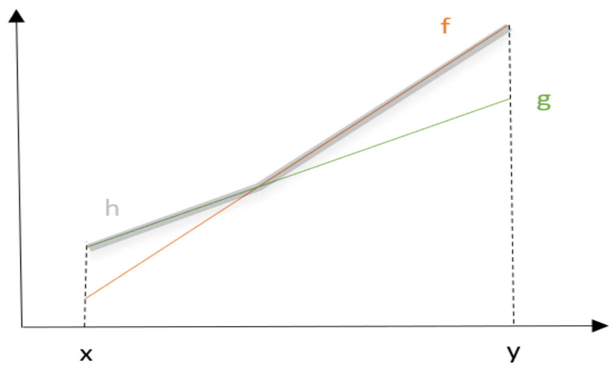
The optimization function is the distance from the origin that we try to maximize and the inequality constraint is for all the points in the ball.

We can easily form the hessian matrix of the objective function:

$$Hf = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Which is a positive definite matrix so the objective function is convex and the statement is not True.

c) We first plot the functions f and g and check the validity of the statement on the figure. Then we provide a proof for the statement. In the following figure we have f (orange), g(green) and h (shady one) which is max $\{f, g\}$.




Left side of inequality: The maximum value of the statement depends on the value of $\lambda$. If $\lambda$ is small enough:

$$\max\{\lambda f(x) + (1-\lambda)f(y), \lambda g(x) + (1-\lambda)g(y)\} = \lambda g(x) + (1-\lambda)g(y)$$

Otherwise

$$\max\{\lambda f(x) + (1-\lambda)f(y), \lambda g(x) + (1-\lambda)g(y)\} = \lambda f(x) + (1-\lambda)f(y)$$

In either cases, h is the maximum of f and g **at any point** and

$$\lambda h(x) + (1-\lambda)h(y) \geq \max\{\lambda f(x) + (1-\lambda)f(y), \lambda g(x) + (1-\lambda)g(y)\}$$

so the statement is true.

Proof:

It is easy to prove that:

$$\lambda.\max\{f(x), g(x)\} + (1-\lambda).\max\{f(y), g(y)\} \geq \lambda f(x) + (1-\lambda)f(y)$$

And similarly

$$\lambda.\max\{f(x), g(x)\} + (1-\lambda).\max\{f(y), g(y)\} \geq \lambda g(x) + (1-\lambda)g(y)$$

By combining them, we have

$$\lambda.\max\{f(x), g(x)\} + (1-\lambda).\max\{f(y), g(y)\}$$
$$\geq max\{\lambda f(x) + (1-\lambda)f(y), \lambda g(x) + (1-\lambda)g(y)\}$$

d) For convex functions we have:

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$$

And for concave functions:

$$f(\theta x + (1-\theta)y) \geq \theta f(x) + (1-\theta)f(y)$$

So if the function f is convex and concave at the same time then:

$$f(\theta x + (1-\theta)y) = \theta f(x) + (1-\theta)f(y) \ (1)$$

Without loss of generality, we set $g(x) = f(x) - f(0)$, So g has no constant value and $g(0) = 0$. Now we prove that

1- $g(\alpha x) = \alpha g(x)$ and
2- $g(x+y) = g(x) + g(y)$

We use the property 1 :

$$g(\alpha x) = g(\alpha x + (1-\alpha) * 0) = \alpha g(x) + (1-\alpha)g(0) = \alpha g(x)$$
$$g(x+y) = g\left(\frac{1}{2}.2x + \frac{1}{2}.2y\right) = \frac{1}{2}g(2x) + \frac{1}{2}g(2y) = g(x) + g(y)$$

So g is an affine function in the form of $g(x) = Ax$ and we can derive f by simply adding a constant to g:

$$f(x) = g(x) + f(0) = Ax + b$$

**Question 5:**

**a)**

We form the Hessian matrix of the function u:

$$u(f,c,s) = \ln(f+1) + \ln(c+1) + \ln(s+1)$$

So

$$H_u = \begin{bmatrix} \frac{\partial^2 u}{\partial f^2} & \frac{\partial^2 u}{\partial f \partial c} & \frac{\partial^2 u}{\partial f \partial s} \\ \frac{\partial^2 u}{\partial c \partial f} & \frac{\partial^2 u}{\partial c^2} & \frac{\partial^2 u}{\partial c \partial s} \\ \frac{\partial^2 u}{\partial f \partial s} & \frac{\partial^2 u}{\partial s \partial c} & \frac{\partial^2 u}{\partial s^2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{(f+1)^2} & 0 & 0 \\ 0 & -\frac{1}{(c+1)^2} & 0 \\ 0 & 0 & -\frac{1}{(s+1)^2} \end{bmatrix}$$

Which is obviously negative definite for all the $(f,c,s)$ in the domain, so the function u is strictly concave.

b)

So the time of the father is allocated to three different tasks:

$$s + c + h_w = 24$$

Where $h_w$ is the working time. Moreover the income should be equal to the expenses so:

$$h_w . w \geq f$$

So over all

$$(24 - s - c).w - f \geq 0$$

c)

The feasible set is:

$$g(f,c,s) = (24 - s - c).w - f \geq 0$$

$$0 \leq s \leq 24$$

$$0 \leq c \leq 24$$

$$0 \leq f \leq 24w$$

d)

$$\max_{f,c,s} \ln(f+1) + \ln(c+1) + \ln(s+1)$$

$$st. (24 - s - c).w - f \geq 0$$

$$s, c, f \geq 0$$

We form the Lagrangian:

$$L = \ln(f+1) + \ln(c+1) + \ln(s+1) + \lambda((24 - s - c).w - f)$$

The first order conditions:

$$\frac{\partial L}{\partial f}=\frac{1}{f+1}-\lambda=0$$

$$\frac{\partial L}{\partial c}=\frac{1}{c+1}-\lambda w=0$$

$$\frac{\partial L}{\partial s}=\frac{1}{s+1}-\lambda w=0$$

$$\lambda[(24-s-c).w-f]=0$$

e,f)

from the first three conditions

$$\lambda=\frac{1}{f+1}=\frac{1}{w(c+1)}=\frac{1}{w(s+1)}\rightarrow c=s=\frac{f+1}{w}-1$$

since $\lambda\neq 0$, the budget constraint should bind so

$$(24-s-c).w-f=0$$

using these two constraints, we have:

$$(24-2c)w=w(c+1)-1\rightarrow 3cw=23w+1$$

$$c^*=s^*=\frac{23w+1}{3w}$$

$$f^*=w\left(\frac{23w+1}{3w}+1\right)-1=\frac{26w}{3}-\frac{2}{3}$$

The Weierstrass' theorem is satisfied because the objective function is a continuous function that is defined on a closed interval.