Mathematics for Economists: Lecture 10

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This lecture covers

1. The value function
2. Interpreting the Lagrange multipliers
3. Duality in consumer choice
4. Value functions for profit maximizing firms

The value function: motivating example

- Let's begin with a concrete example.
- A profit maximizing monopolist firm selling in a market where the market demand curve is given by $p = a - bq$ with $a, b > 0$.
- The cost of producing q units is cq with $0 < c < a$.
- The profit is equal to revenue net of cost, i.e. the solves

$$\max_{q \geq 0} (a - bq)q - cq.$$

The value function

- Since profit at $q = 0$ is zero, the inequality constraint is not binding.
- Since the objective function is strictly concave, any point satisfying the first-order condition is an optimum.
- Setting the derivative of the objective function with respect to q at zero gives:

$$q^* = \frac{a - c}{2b}.$$

- The maximum profit that the firm can get is therefore

$$\pi(a, b, c) = (a - c)q^* - b(q^*)^2 = \frac{(a - c)^2}{4b}.$$

The value function

- How does the optimal profit depend on the parameters a, b, c? Since the optimal q changes, maybe this is quite complicated?
- Just take the derivatives of the function $\pi(a, b, c)$ with respect to its variables.
- We see that $\frac{\partial \pi(a,b,c)}{\partial a} = \frac{a-c}{2b} = -\frac{\partial \pi(a,b,c)}{\partial c}$, and $\frac{\partial \pi(a,b,c)}{\partial b} = -\frac{(a-c)^2}{4b^2}$.
- But it is also true that $\frac{\partial \pi(a,b,c)}{\partial a} = q^* = -\frac{\partial \pi(a,b,c)}{\partial c}$ and $\frac{\partial \pi(a,b,c)}{\partial b} = -(q^*)^2$.
- But these are the partial derivatives of the objective function with respect to each parameter
- Is there a reason behind this or is this just a coincidence?

The value function

- Consider an unconstrained maximization problem of a function of a single real variable x, where the objective function depends on a parameter $\alpha \in \mathbb{R}$.

$$\max_{x \in \mathbb{R}} f(x, \alpha).$$

- Let $x(\alpha)$ be the solution to this problem.
- Consider the maximum value of the objective function that is achievable at the exogenous variable (or parameter) $\hat{\alpha}$, i.e. $f(x(\hat{\alpha}), \hat{\alpha})$.
- We call this new function the *value function* of the problem and denote

$$V(\alpha) := f(x(\alpha), \alpha).$$

The value function

- At the (unconstrained) optimum $x(\hat{\alpha})$, by the first-order condition:

$$\frac{\partial f(x(\hat{\alpha}), \hat{\alpha})}{\partial x} = 0.$$

Suppose that f is twice continuously differentiable and that the second order condition is satisfied so that

$$\frac{\partial^2 f(x(\hat{\alpha}))}{\partial x^2} < 0.$$

- Then we can use implicit function theorem to see that $x(\alpha)$ satisfying the first-order condition exists in some neighborhood of $\hat{\alpha}$.

The value function

- By chain rule:

$$V'(\hat{\alpha}) = \frac{\partial f(x(\hat{\alpha}), \hat{\alpha})}{\partial x} x'(\hat{\alpha}) + \frac{\partial f(x(\hat{\alpha}), \hat{\alpha})}{\partial \alpha}.$$

- Since $\frac{\partial f(x(\hat{\alpha}),\hat{\alpha})}{\partial x} = 0$, we get

$$V'(\hat{\alpha}) = \frac{\partial f(x(\hat{\alpha}), \hat{\alpha})}{\partial \alpha}.$$

Envelope theorem

- Envelope theorem states that for twice continuously differentiable functions $f(x, \alpha)$, and the value function $V(\alpha) = \max_x f(x, \alpha)$, we have

$$V'(\alpha) = \frac{\partial f(x(\alpha), \alpha)}{\partial \alpha}.$$

- In words, when a parameter changes, the maximum value of the problem changes only through the direct effects on the objective function.
- Indirect effects on the value vanish because of the first-order condition on x.
- Can you relate the theorem to the motivating example?

Envelope theorem

- In the more general case, where $\boldsymbol{x} \in \mathbb{R}^n$, the message is exactly the same. The first order-condition is now:

$$\frac{\partial f(\boldsymbol{x}(\hat{\alpha}), \hat{\alpha})}{\partial x_i} = 0 \text{ for all } i \in \{1, ..., n\}.$$

- Assuming the conditions for implicit function theorem, we have by chain rule:

$$V'(\hat{\alpha}) = \sum_{i=1}^{n} \frac{\partial f(\boldsymbol{x}(\hat{\alpha}), \hat{\alpha})}{\partial x_i} x_i'(\hat{\alpha}) + \frac{\partial f(\boldsymbol{x}(\hat{\alpha}), \hat{\alpha})}{\partial \alpha}.$$

- Again, the first term vanishes by first-order condition and we are left with

$$V'(\hat{\alpha}) = \frac{\partial f(\boldsymbol{x}(\hat{\alpha}), \hat{\alpha})}{\partial \alpha}.$$

Envelope theorem in equality constrained problems

- Suppose that we have an equality constrained parametric maximization problem for $\boldsymbol{x} \in \mathbb{R}^n$:

$$\max_{\boldsymbol{x}} f(\boldsymbol{x}, \alpha)$$

$$\text{subject to } g(\boldsymbol{x}, \alpha) = 0.$$

- Let $\boldsymbol{x}(\alpha)$ denote the optimal solution and assume sufficient differentability that we can use implicit function theorem around the solution as before. (I.e. assume that the objective function is twice continuously differentiable).
- The value function is still defined as: $V(\alpha) = f(\boldsymbol{x}(\alpha), \alpha)$.
- Form the Lagrangean:

$$\mathcal{L}(\boldsymbol{x}, \mu; \alpha) = f(\boldsymbol{x}, \alpha) - \mu g(\boldsymbol{x}, \alpha).$$

Envelope theorem in equality constrained problems

Theorem (Envelope theorem)

In an optimization problem subject to an equality constraint, we have:

$$V'(\alpha) = \frac{\partial \mathcal{L}(\boldsymbol{x}, \mu; \alpha)}{\partial \alpha}.$$

Proof.

$$V'(\hat{\alpha}) = \sum_{i=1}^{n} \frac{\partial f(\boldsymbol{x}(\hat{\alpha}), \hat{\alpha})}{\partial x_i} x_i'(\hat{\alpha}) + \frac{\partial f(\boldsymbol{x}(\hat{\alpha}), \hat{\alpha})}{\partial \alpha}.$$

Now the first-order condition implies that

$$\frac{\partial f(\boldsymbol{x}(\hat{\alpha}), \hat{\alpha})}{\partial x_i} = \mu \frac{\partial g(\boldsymbol{x}(\hat{\alpha}), \hat{\alpha})}{\partial x_i}.$$

Envelope theorem in equality constrained problems

Since the constraint holds for all α, we have

$$\sum_{i=1}^{n} \frac{\partial g(\boldsymbol{x}(\hat{\alpha}), \hat{\alpha})}{\partial x_i} x_i'(\hat{\alpha}) = -\frac{\partial g(\boldsymbol{x}(\hat{\alpha}), \hat{\alpha})}{\partial \alpha}.$$

Combining these gives:

$$V'(\hat{\alpha}) = \frac{\partial f(\boldsymbol{x}(\hat{\alpha}), \hat{\alpha})}{\partial \alpha} - \mu \frac{\partial g(\boldsymbol{x}(\hat{\alpha}), \hat{\alpha})}{\partial \alpha}.$$

□

Interpreting the Lagrange multipliers

- Envelope theorem gives us a nice way of understanding the Lagrange multipliers in UMP
- The Lagrangean for the UMP with a single binding equality constraint is:

$$\mathcal{L}(\boldsymbol{x}, \lambda) = u(\boldsymbol{x}) - \mu \left[\sum_{i=1}^{n} p_i x_i - w \right].$$

- The maximum value function

$$v(\boldsymbol{p}, w) = \max u(\boldsymbol{x}) \text{ subject to } \boldsymbol{p} \cdot \boldsymbol{x} = w,$$

is called the indirect utility function. It computes the optimal utility level for all combinations of prices $\boldsymbol{p} \in \mathbb{R}^n_{++}$ and income $w > 0$.

Interpreting the Lagrange multipliers

- Envelope theorem tells us that

$$\frac{\partial v(\boldsymbol{p}, w)}{\partial w} = \mu.$$

- This means that if your income is increased by one unit, your maximal utility increases the amount given by the multiplier.
- By reducing income dw you lose μdw of utility and this is why the multiplier is sometimes called the shadow price of income.
- Note also that the usual first-order condition requires:

$$\mu dw = \frac{\partial u(\boldsymbol{x})}{\partial x_i} \frac{dw}{p_i} \text{ for all } i.$$

- With dw of additional income, you can buy $\frac{dw}{p_i}$ units of good i.

Interpreting the Lagrange multipliers

- Envelope theorem also tells us that:

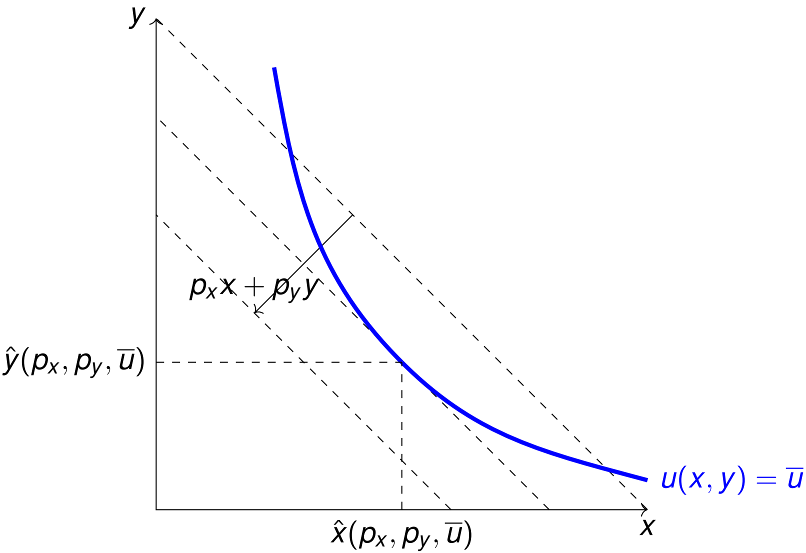
$$\frac{\partial v(\boldsymbol{p}, w)}{\partial p_i} = -\mu x_i.$$

- Combining these two, we have Roy's identity:

$$x_i(\boldsymbol{p}, w) = -\frac{\frac{\partial v(\boldsymbol{p},w)}{\partial p_i}}{\frac{\partial v(\boldsymbol{p},w)}{\partial w}}.$$

- In other words, if you have an indirect utility function, you can compute the demand function by simple partial differentiation.
- In later courses, you will learn what properties on $v(\boldsymbol{p}, w)$ guarantee that it is the indirect utility function of some UMP for some $u(\boldsymbol{x})$.

Figure: Expenditure minimization



Expenditure minimization

- Consider next the expenditure minimization problem from Lecture 9.

$$\min_{\boldsymbol{h} \in X} \boldsymbol{p} \cdot \boldsymbol{x} = \sum_{i=1}^{n} p_i h_i,$$

  subject to

$$u(\boldsymbol{h}) = \overline{u}.$$

- Denote the solution to this problem by $h(\boldsymbol{p}, \overline{u})$. We call $h_i(\boldsymbol{p}, \overline{u})$ the Hicksian or compensated demand for good i.
- The (minimum) value function of this problem $e(\boldsymbol{p}, \overline{u}) = \sum_{i=1}^{n} p_i h_i(\boldsymbol{p}, \overline{u})$ is called the *expenditure function*.
- The objective function is linear in $\boldsymbol{p}$ and hence by the results in Lecture 6, we know that $e(\boldsymbol{p}, \overline{u})$ is concave in $\boldsymbol{p}$.
- Therefore the Hessian matrix of $e(\boldsymbol{p}, \overline{u})$ is negative semidefinite.

Value function for expenditure minimization

- The Lagrangean for interior solutions:

$$\mathcal{L}(\boldsymbol{h}, \mu) = \sum_{i=1}^{n} p_i h_i - \mu(\overline{u} - u(\boldsymbol{h})).$$

- Envelope theorem tells us that

$$\frac{\partial e(\boldsymbol{p}, \overline{u})}{\partial p_i} = h_i.$$

- The partial derivatives of $h_i(\boldsymbol{p}, \overline{u})$ with respect to p_j are the elements of the Hessian matrix of $e(\boldsymbol{p}, \overline{u})$.

Connecting expenditure minimization and UMP

- Hold prices $\hat{p}$ fixed for a moment and ask how high utility you can achieve with income w. The answer is given by the indirect utility function $v(\hat{\boldsymbol{p}}, w)$.
- Ask next what is the minimum expenditure that you must use to achieve utility $v(\hat{\boldsymbol{p}}, w)$.
- By choosing $h_i = x_i(\boldsymbol{p}, w)$ you achieve that utility at expenditure w.
- If you could achieve $v(\boldsymbol{p}, w)$ at a strictly lower cost, then you could achieve a higher utility at $(\boldsymbol{p}, w)$ contradicting the definition of $v(\boldsymbol{p}, w)$.
- So we conclude:

$$e(\hat{\boldsymbol{p}}, v(\hat{\boldsymbol{p}}, w)) = w.$$

Connecting expenditure minimization and UMP

- Similarly, it costs you $e(\hat{\boldsymbol{p}}, \overline{u})$ to reach utility $\overline{u}$.
- If your budget is $e(\hat{\boldsymbol{p}}, \overline{u})$, then you can reach utility level $\bar{u}$ by choosing $x_i = h_i(\boldsymbol{p}, \bar{u})$.
- If you could reach a strictly higher utility level, then by continuity of $u(\cdot)$, you could reach $\bar{u}$ even if you reduced some consumption a bit contradicting the definition of $e(\boldsymbol{p}, \bar{u})$.
- We conclude:

$$\overline{u} = v(\hat{\boldsymbol{p}}, e(\hat{\boldsymbol{p}}, \overline{u})).$$

Figure: UMP for $w = e(\boldsymbol{p}, v(\boldsymbol{p}, w))$

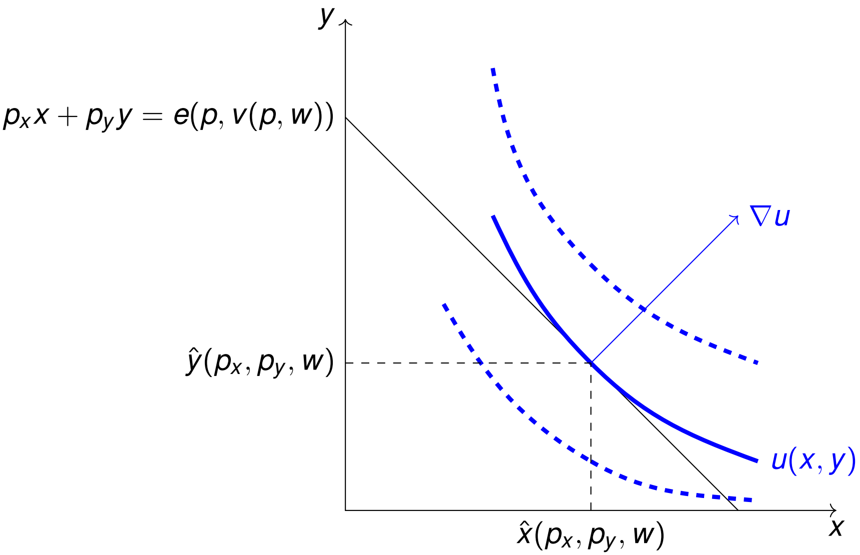
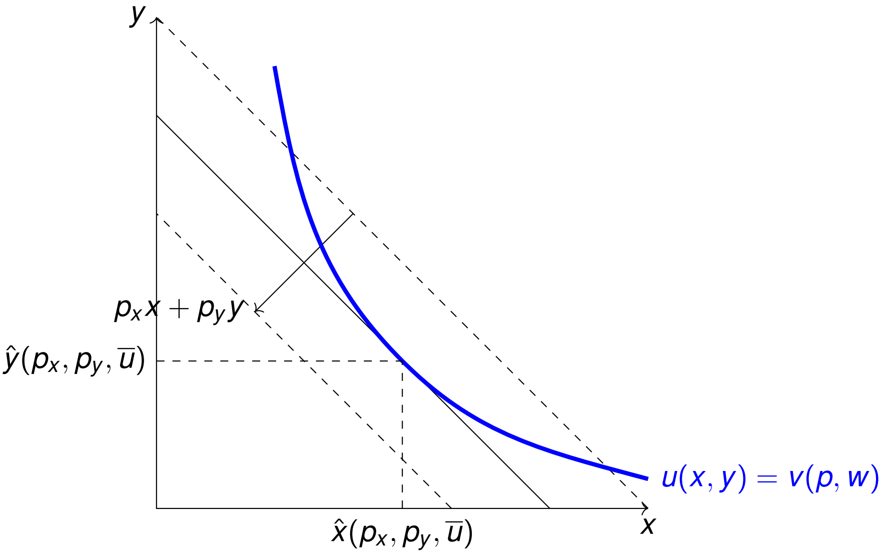


Figure: Expenditure minimization for $\overline{u} = v(\boldsymbol{p}, e(\boldsymbol{p}, \bar{u}))$



Connecting expenditure minimization and UMP

- You can also see that for $\bar{u} = v(\boldsymbol{p}, e(\boldsymbol{p}, \bar{u})$ and $e(\boldsymbol{p}, v(\boldsymbol{p}, w)) = w$ the solutions to expenditure minimization and UMP coincide for all p:

$$h_i(\boldsymbol{p}, \bar{u}) = x_i(\boldsymbol{p}, e(\boldsymbol{p}, \bar{u})) \text{ for all } i,$$

$$h_i(\boldsymbol{p}, v(\boldsymbol{p}, w)) = x_i(\boldsymbol{p}, w) \text{ for all } i.$$

Connecting expenditure minimization and UMP

- Differentiate the first of these identities with respect to p_j to get:

$$\begin{aligned}
\frac{\partial h_i(\boldsymbol{p}, \overline{u})}{\partial p_j} &= \frac{\partial x_i(\boldsymbol{p}, w)}{\partial p_j} + \frac{\partial x_i(\boldsymbol{p}, w)}{\partial w} \frac{\partial e(\boldsymbol{p}, \overline{u})}{\partial p_j} \\
&= \frac{\partial x_i(\boldsymbol{p}, w)}{\partial p_j} + \frac{\partial x_i(\boldsymbol{p}, w)}{\partial w} h_j(\boldsymbol{p}, \overline{u}) \\
&= \frac{\partial x_i(\boldsymbol{p}, w)}{\partial p_j} + \frac{\partial x_i(\boldsymbol{p}, w)}{\partial w} x_j(\boldsymbol{p}, e(\boldsymbol{p}, \overline{u})) \\
&= \frac{\partial x_i(\boldsymbol{p}, w)}{\partial p_j} + \frac{\partial x_i(\boldsymbol{p}, w)}{\partial w} x_j(\boldsymbol{p}, w).
\end{aligned}$$

Connecting expenditure minimization and UMP

- This is the famous Slutsky equation for income and substitution effects.
- The observable change in Marshallian $\frac{\partial x_i(\boldsymbol{p},w)}{\partial p_j}$ demands can be decomposed into a substitution effect, i.e. the change in compensated demand $\frac{\partial h_i(\boldsymbol{p},\overline{u})}{\partial p_j}$ and the observable income effect $\frac{\partial x_i(\boldsymbol{p},w)}{\partial w} x_j(\boldsymbol{p}, w)$.

$$\frac{\partial x_i(\boldsymbol{p}, w)}{\partial p_j} = \frac{\partial h_i(\boldsymbol{p}, \overline{u})}{\partial p_j} - \frac{\partial x_i(\boldsymbol{p}, w)}{\partial w} x_j(\boldsymbol{p}, w).$$

- Since we know that the Hessian of $e(\boldsymbol{p}, \overline{u})$ is negative definite, we know that its diagonal elements are non-positive.
- Hence the effect of increasing p_i on x_i is negative whenever the demand for i is increasing in income (we say then that i is a non-inferior good).

Conditions for demand functions

- We have seen up to now that demand functions arising from utility maximization problems satisfy:
  1. Homogeneity of degree 0 (budget set does not change if all prices and income multiplied by the same strictly positive number).
  2. If the utility function is strictly increasing, then all income is used:

$$\sum_{i=1}^{n} p_i x_i(\boldsymbol{p}, w) = w \text{ for all } \boldsymbol{p}, w > 0.$$

  3. The matrix X (called the *Slutsky matrix*) with $(i,j)^{th}$ element $x_{ij} = \frac{\partial x_i(\boldsymbol{p},w)}{\partial p_j} + \frac{\partial x_i(\boldsymbol{p},w)}{\partial w} x_j(\boldsymbol{p}, w)$ is negative semidefinite.
  4. From Young's theorem, Slutsky matrix is symmetric.

Conditions for demand functions

- We could ask conversely, what conditions on a vector valued function $\boldsymbol{x}(\boldsymbol{p}, w)$ guarantee that it is the Marshallian demand for some utility maximization problem.
- A remarkable (but unfortunately somewhat hard to prove) result states that the above four conditions are sufficient.
- In other words: Any vector valued function that is homogenous of degree 0, uses the entire budget and has symmetric and positive definite Slutsky matrix is a demand function for some (strictly increasing and quasiconcave) utility function.

Cost function of a competitive firm

- A firm chooses its inputs k, l to minimize the cost of reaching a production target of $\overline{q}$ at given input prices r, w.
- The production function is assumed to be a strictly increasing and quasiconcave function $f(k, l)$.

$$\min_{(k,l)\in\mathbb{R}^2_+} rk + wl$$

subject to

$$f(k, l) = q.$$

Cost function of a competitive firm

- The value function of this problem is called the *cost function* of the firm and denoted by $c(r, w, q)$.

$$c(r, w, q) = rk(r, w, q) + wl(r, w, q),$$

where $k(r, w, q), l(r, w, q)$ solve the cost minimization problem.

- These are called the conditional factor demands.
  As in the case with expenditure minimization, we see that the cost function is concave in r, w since it is the minimum of linear functions of r, w.
- Therefore the Hessian of the cost function is negative semidefinite.

Cost function of a competitive firm

- By envelope theorem, we have the result known as Shephard's lemma:

$$\frac{\partial c(r,w,q)}{\partial r} = k(r,w,q), \qquad \frac{\partial c(r,w,q)}{\partial w} = l(r,w,q).$$

- Negative semidefiniteness of the Hessian of c implies that (since the diagonal elements must be non-positive)

$$\frac{\partial k(r,w,q)}{\partial r} \leq 0, \frac{\partial l(r,w,q)}{\partial w} \leq 0.$$

- In words, conditional factor demands are decreasing in own price (not surprisingly).

Profit function of a competitive firm

- We end this part of the course with the analysis of profit maximization for a price taking firm.
- There are two ways to think about this. Either minimize cost for each production level q to get $c(r, w, q)$ and then choose q optimally to maximize $pq - c(r, w, q)$, where p is the price of the output.
- Alternatively, you can write directly:

$$\max_{k,l,q} pq - rk - wl$$

subject to

$$q = f(k, l).$$

Profit function of a competitive firm

- An advantage of the second approach is that the problem is immediately seen to be linear in the input and output prices p, r, w.
- Let $q(p, r, w), k(p, r, w), l(p, r, w)$ be the optimal output and input choices in the problem. The value function $\pi(p, r, w)$ is called the profit function of the firm.
- Since π is the maximum of linear functions in p, r, w, we get by Lecture 6 that π is convex and hence its Hessian is positive semidefinite.

Profit function of a competitive firm

- The envelope theorem gives us Hotelling's lemma:

$$\frac{\partial \pi(p,r,w)}{\partial p} = q(p,r,w), \quad \frac{\partial \pi(p,r,w)}{\partial r} = -k(p,r,w), \quad \frac{\partial \pi(p,r,w)}{\partial w} = -l(p,r,w).$$

- Since π is positive semi-definite, its diagonal elements are non-negative.
- This gives the 'Law of Supply' (supply increases in output price)

$$\frac{\partial q(p,r,w)}{\partial p} \geq 0,$$

and the 'Law of Factor Demands' (factor demand decrease in factor price):

$$\frac{\partial k(p,r,w)}{\partial r} \leq 0, \quad \frac{\partial l(p,r,w)}{\partial w} \leq 0.$$

Profit function of a competitive firm

- As you can see, the theory of the competitive firm is easier than consumer theory since changes in prices do not change the constraint set (as with the budget set)
- You will see the firm's problem in some form in almost all branches of economics and in particular in Intermediate Microeconomics
- Of course, in many industries firms are not competitive $\rightarrow$ Industrial organization
- Firms do not make decisions but people do and people may have different objectives $\rightarrow$ Organizational economics, Contract theory