Mathematics for Economists: Lecture 9

Juuso Välimäki

Aalto University School of Business

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This lecture covers

1. Economic applications of constrained optimization
   1.1 Utility maximization continued
   1.2 Expenditure and cost minimization
2. First look at duality and value functions

Utility maximization: Cobb-Douglas utility function

- Perhaps the most used functional form in economics is the Cobb-Douglas function
$$u(x) = x^{\alpha} y^{1-\alpha},$$
for some $\alpha \in (0, 1)$.
- The distinguishing feature of this form is that the function is homogenous of degree 1.
- You can check with the Hessian matrix (as an exercise) that $u(x, y)$ is concave and therefore also quasiconcave.

Utility maximization: Cobb-Douglas utility function

- Both marginal utilities are strictly positive at all $(x, y) > (0, 0)$ and

$$\lim_{x \to 0} \frac{\partial u(x, \bar{y})}{\partial x} = \lim_{y \to 0} \frac{\partial u(\bar{x}, y)}{\partial y} = \infty,$$

  for $\bar{x}, \bar{y} > 0$. Since $x = y = \epsilon$ is feasible for small enough ϵ and $u(\epsilon, \epsilon) = \epsilon > 0 = u(0, 0)$, we know that even though $(0, 0, 0, 0, 0)$ is a critical point of the utility function, it is not a maximum.
- Similarly $(0, y)$ and $((x, 0)$ cannot be optimal solutions and we can restrict to interior points.

UMP: Cobb-Douglas utility function

- The requirement that $MRS_{x,y} = \frac{p_x}{p_y}$ gives:

$$\frac{\alpha y}{(1-\alpha)x} = \frac{p_x}{p_y} \text{ or } p_x x = \frac{\alpha}{1-\alpha} p_y y.$$

- Using the budget constraint:

$$p_x x + p_y y = w,$$

we get:

$$x(p_x, p_y, w) = \frac{\alpha w}{p_x}, \text{ and } \qquad y(p_x, p_y, w) = \frac{(1-\alpha) w}{p_y}.$$

UMP: Cobb-Douglas utility function

- For the Cobb-Douglas utility function, you get the result that the expenditure shares $\frac{p_x x}{w} = \alpha$ and $\frac{p_y y}{w} = 1 - \alpha$ do not depend on prices or w.
- This extends easily to the case with n goods and $u(x) = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ with $\alpha_i > 0, \sum_i \alpha_i = 1$ at prices $p = (p_1, ..., p_n)$. Then you have:

$$x_i(p, w) = \frac{\alpha_i w}{p_i}.$$

- This is not very realistic.
- The rich and the poor use their budgets very differently.

UMP: Stone-Geary utility function

- One way to get more realistic consumption patters is to define the utility function for consumptions above a level needed for subsistence.
- Let $\underline{\boldsymbol{x}} = (\underline{x_1}, ..., \underline{x_n})$ be the levels of each good needed for survival and assume that $w \geq \boldsymbol{p} \cdot \underline{\boldsymbol{x}}$.
- The utility function for $\boldsymbol{x} \in \mathbb{R}^n$ such that $x_i \geq \underline{x_i}$ is of Cobb-Douglas -like form:

$$u(\boldsymbol{x}) = (x_1 - \underline{x_1})^{\alpha_1} \dots (x_n - \underline{x_n})^{\alpha_n},$$

  where $0 < \alpha_i < 1$ for all i and $\sum_{i=1}^{n} \alpha_i = 1$.
- Notice that the marginal utility for good i is infinite if $x_i = \underline{x_i}$ and that the utility function is strictly increasing in all of its components.
- Hence we still have an interior solution and the budget constraint binds.

UMP: Stone-Geary utility function

- We get as above:

$$\frac{\frac{\partial u(\boldsymbol{x})}{\partial x_i}}{\frac{\partial u(\boldsymbol{x})}{\partial x_k}} = \frac{\alpha_i(x_k - \underline{x_k})}{\alpha_k(x_i - \underline{x_i})} = \frac{p_i}{p_k} \text{ for all } i, k,$$

$$\sum_{i=1}^{n} p_i x_i = w.$$

- Taking $k = 1$, we get that

$$x_i - \underline{x_i} = \frac{\alpha_i p_1}{\alpha_1 p_i}(x_1 - \underline{x_1}) \text{ for all } i. \tag{9}$$

- Multiplying both sides by p_i and summing over i gives:

$$\sum_{i=1}^{n} p_i(x_i - \underline{x_i}) = \frac{p_1 \sum_{i=1}^{n} \alpha_i}{\alpha_1}(x_1 - \underline{x_1}).$$

UMP: Stone-Geary utility function

- So we can solve:

$$x_1 - \underline{x_1} = \frac{\alpha_1 (w - \sum_{i=1}^n p_i \underline{x_i})}{p_1},$$

where we used the budget constraint $\sum_{i=1}^n p_i x_i = w$ and $\sum_{i=1}^n \alpha_i = 1$

- By (9), we see that

$$x_i - \underline{x_i} = \frac{\alpha_i (w - \sum_{j=1}^n p_j \underline{x_j})}{p_i}.$$

- The consumer uses a constant fraction of her excess income (above what is needed for the necessities $\underline{x}$) in constant shares given by the α_i.
- Since the poor have less excess wealth, their consumption fractions are closer to the ones given by the subsistence levels $\beta_i := \frac{\underline{x_i}}{\sum_i \underline{x_i}}$.

Quasilinear utility function

- We end the section on utility maximization with $u(x, y) = v(x) + y$, where v is a strictly increasing and strictly concave function subject to non-negativity of x, y and the budget constraint

$$p_x x + y \leq w.$$

- Are we losing generality in assuming that $p_y = 1$?
- Now $MRS_{x,y} = v'(x)$.

Quasilinear utility function

- If $v'(\frac{w}{p_x}) > p_x$, or if $v'(0) < p_x$, then we have a corner solution.
- In the first case, $x(p_x, w) = \frac{w}{p_x}, y(p_x, w) = 0$.
- In the second case, $x(p_x, w) = 0$ and $y(p_x, w) = w$.
- Otherwise $x(p_x, w)$ solves

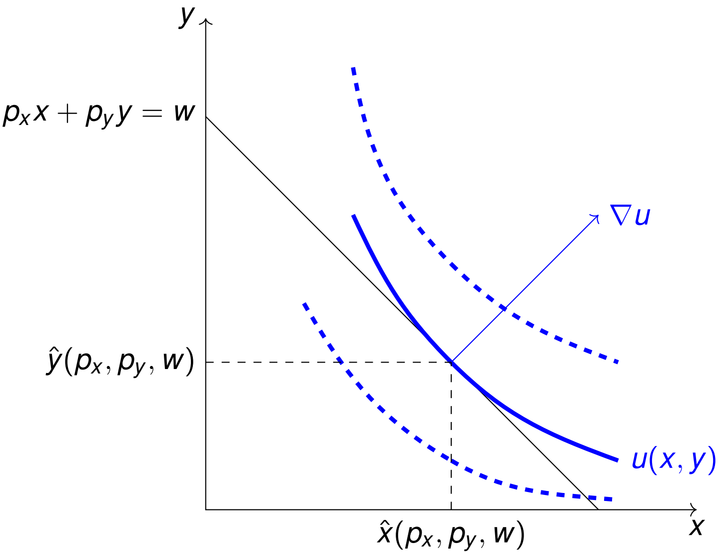
$$v'(x) = p_x,$$

and

$$y = (w - p_x x(p_x, w)).$$

- Notice that $MRS_{x,y}$ does not depend on y. A higher y simply shifts vertically the indifference curves.
- This utility function lies behind partial equilibrium analysis in microeconomics where x is sold in the market of interest and y is everything else.
- y represents expenditure on all other goods or total income. With quasi-linear utility, there are no income effects (as long as we remain in the range for interior solutions).

Figure: Utility maximization problem



Expenditure minimization problem

- Suppose the consumer has a utility function given by $u(x, y)$
- How do you minimize expenditure to reach at least the utility level $\bar{u}$?

$$\min_{x,y} p_x x + p_y y$$

subject to:

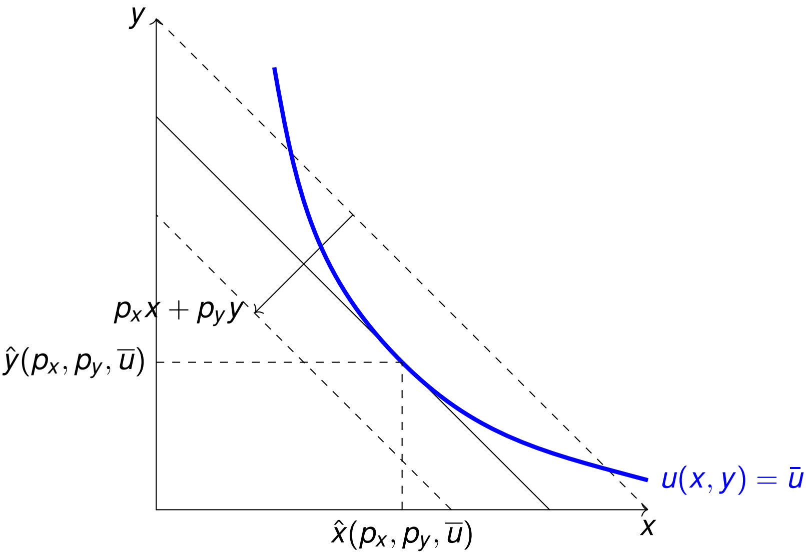
$$x, y \geq 0, \quad u(x, y) \geq \bar{u}.$$

- We'll connect UMP and EMP in the second part of this lecture.
- Lagrangean for the problem:

$$\mathcal{L}(x, y, \lambda_1, \lambda_2, \lambda_3) = p_x x + p_y y - \lambda_1 (u(x, y) - \bar{u}) + \lambda_2 x + \lambda_3 y.$$

- Exercise: What are the Kuhn-Tucker first-order conditions for this problem?

Figure: Expenditure minimization problem



Cost minimization problem for a firm

- A firm chooses its inputs k, l to minimize the cost of reaching a production target of $\overline{q}$ at given input prices r, w.
- Notice that the objective function is quasiconvex.
- The production function is assumed to be a strictly increasing and quasiconcave function $f(k, l)$.

$$\min_{(k,l)\in\mathbb{R}^2_+} rk + wl$$

subject to

$$f(k, l) \geq \bar{q}, k, l \geq 0.$$

Cost minimization problem for a firm

- Notice that the feasible set is closed and convex.
- It is not bounded but is this a problem for existence of a solution?
- Lagrangean for the problem:

$$\mathcal{L}(k, l, \lambda_1, \lambda_2, \lambda_3) = rk + wl - \lambda_1(f(k, l) - \bar{q}) - \lambda_2 k - \lambda_3 l.$$

- It is often assumed that $f(0, l) = f(k, 0) = 0$ and then the non-negativity constraints are not binding. (Of course, with e.g. linear technologies, you must consider corner solutions).

Cost minimization problem for a firm: Cobb-Douglas case

- Let $f(k,l) = k^\alpha l^{1-\alpha}$.
- $(\hat{k}, \hat{l})$ such that $\hat{k} = 0$ or $\hat{l} = 0$ are not in the feasible set.
- I leave it as an exercise to argue that the constraint $f(k,l) \geq \bar{q}$ binds at optimum, i.e.

$$f(\hat{k}, \hat{l}) = \bar{q}.$$

- First-order conditions for optimum are:

$$r = \lambda_1 \alpha \left(\frac{\hat{l}}{\hat{k}}\right)^{1-\alpha}, w = \lambda_1 (1-\alpha) \left(\frac{\hat{k}}{\hat{l}}\right)^{\alpha},$$

and

$$f(\hat{k}, \hat{l}) = \bar{q}.$$

Cost minimization problem for a firm: Cobb-Douglas case

- From the first two, you get:

$$\frac{r}{w} = \frac{\alpha}{1-\alpha}\frac{\hat{l}}{\hat{k}}.$$

- Solving for $\hat{k}$ and substituting into the constraint gives:

$$\hat{k} = \bar{q}\left(\frac{\alpha w}{(1-\alpha)r}\right)^{1-\alpha}, \hat{l} = \bar{q}\left(\frac{(1-\alpha)r}{\alpha w}\right)^{\alpha}.$$

- You can also verify that $\hat{\lambda}_1 > 0$.
- The minimal cost for achieving production level $\bar{q}$ is

$$c(\bar{q}; r, w) = r\hat{k} + w\hat{l} = \bar{q}(\alpha)^{-\alpha}(1-\alpha)^{\alpha-1}r^{\alpha}w^{1-\alpha}.$$

Comparative statics of utility maximization

- Recall from Principles of Economics I,
  1. Substitution effect of price changes
  2. Income effect of price changes
- We will see how to express these mathematically by connecting utility maximization and expenditure minimization problems.
- In order to be able to do this, we need to understand the value functions of the two problems.

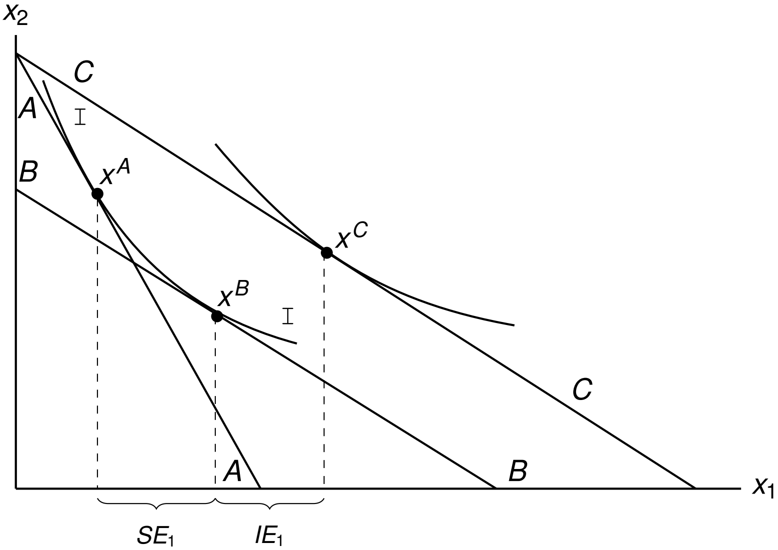


Figure: Hicks decomposition

Value function: utility maximization

- What is the highest utility level that a consumer can reach when maximizing her utility subject to a budget constraint?
- If $(x_1(p_1, ..., p_n, w), ..., x_n(p_1, ..., p_n, w))$ is her optimal demand, we get the utility level by plugging the demand back into the utility function:

$$u(x_1(p_1, ..., p_n, w), ..., x_n(p_1, ..., p_n, w)).$$

- Notice that this maximized value is a function of the exogenous variables $(\boldsymbol{p}, w)$. We call it the value function of the problem.
- For utility maximization problems, the value function is called the *indirect utility function*:

$$v(p_1, ..., p_n, w) := u(x_1(p_1, ..., p_n, w), ..., x_n(p_1, ..., p_n, w)).$$

Value function: expenditure minimization

- Let's return to the expenditure minimization problem:

$$\min_{\boldsymbol{h} \in X} \boldsymbol{p} \cdot \boldsymbol{h} = \sum_{i=1}^{n} p_i h_i,$$

subject to

$$u(\boldsymbol{h}) = \overline{u}.$$

- Denote the solution to this problem by $\boldsymbol{h}(\boldsymbol{p}, \overline{u})$. We call $h_i(\boldsymbol{p}, \overline{u})$ the Hicksian or compensated demand for good i.
- The *value function* of this problem is the minimal expenditure needed to achieve utility level $\bar{u}$:

$$e(\boldsymbol{p}, \overline{u}) = \sum_{i=1}^{n} p_i h_i(\boldsymbol{p}, \overline{u}).$$

Connecting expenditure minimization and UMP

- Hold prices $\hat{\boldsymbol{p}}$ fixed for a moment and ask how high utility you can achieve with income w. The answer is given by the indirect utility function $v(\hat{\boldsymbol{p}}, w)$.
- Ask next what is the minimum expenditure that you must use to achieve utility $v(\hat{\boldsymbol{p}}, w)$. The following figures should convince you that for all $\hat{\boldsymbol{p}}$,

$$e(\hat{\boldsymbol{p}}, v(\hat{\boldsymbol{p}}, w)) = w.$$

- It costs you $e(\hat{\boldsymbol{p}}, \overline{u})$ to reach utility $\overline{u}$. If your budget is $e(\hat{\boldsymbol{p}}, \overline{u})$, then the maximal utility that you can reach is for all $\hat{\boldsymbol{p}}$,

$$\overline{u} = v(\hat{\boldsymbol{p}}, e(\hat{\boldsymbol{p}}, \overline{u})).$$

Figure: UMP for $w = e(p, v(p, w))$

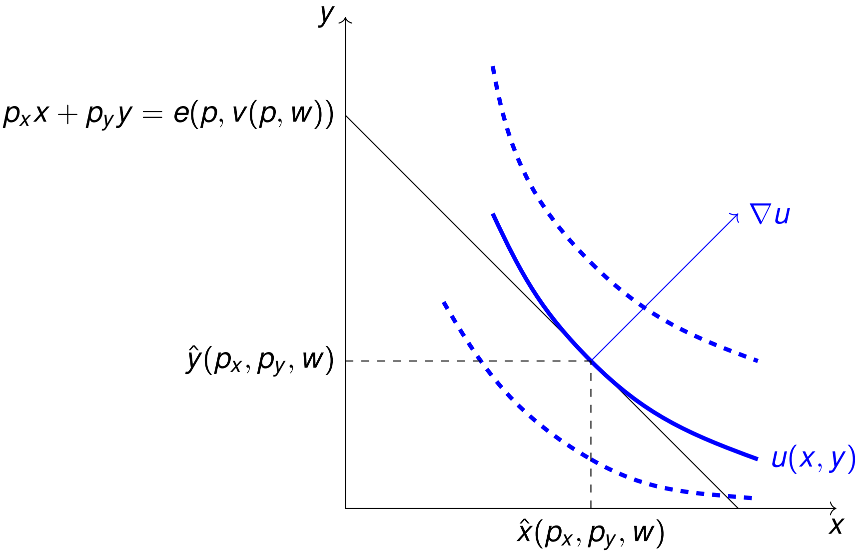
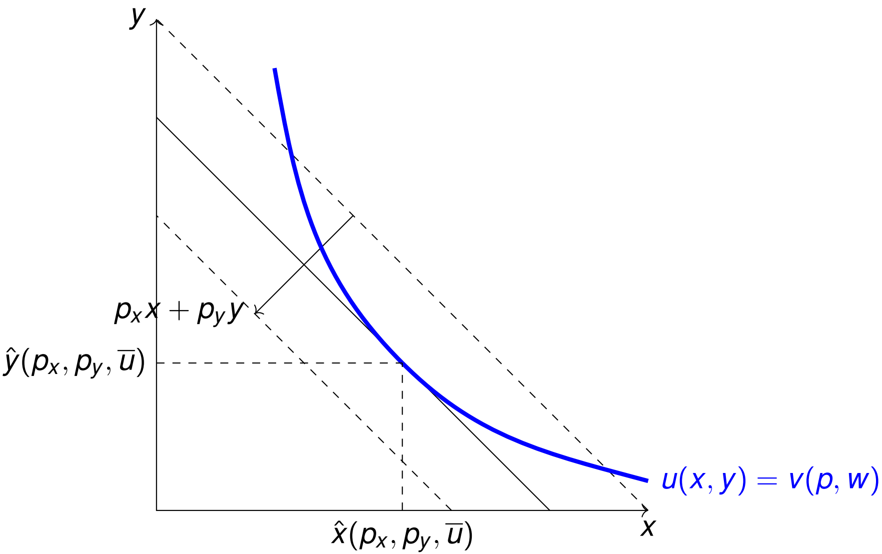


Figure: Expenditure minimization for $\overline{u} = v(p, w)$



Connecting expenditure minimization and UMP

- You can also see that for $\bar{u} = v(\boldsymbol{p}, e(\boldsymbol{p}, \bar{u})$ and $e(\boldsymbol{p}, v(\boldsymbol{p}, w)) = w$ the solutions to expenditure minimization and UMP coincide for all $\boldsymbol{p}$:

$$h_i(\boldsymbol{p}, \bar{u}) = x_i(\boldsymbol{p}, e(\boldsymbol{p}, \bar{u})) \text{ for all } i,$$

$$h_i(\boldsymbol{p}, v(\boldsymbol{p}, w)) = x_i(\boldsymbol{p}, w) \text{ for all } i.$$