Solutions to the problem set 5:

Question 1:

a) Matrix A is non-singular if and only if its determinant is not zero. Considering this, it is obvious that the statement is not correct. One easy example would be:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\det(A) = 1 \text{ so A is non-singular}$$

We can easily calculate the eigen-values as follows:

$$\det(A - \lambda I) = 0 \rightarrow det \begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} = 0$$

$$\rightarrow (1-\lambda)^2 = 0 \rightarrow \lambda_1 = \lambda_2 = 1$$

None of the eigen-values is zero.

b) We have the following linear system of equations:

$$Ax = b$$

Assuming that x_1, x_2 solve this, we have

$$Ax_1 = b \; and \; Ax_2 = b$$

$$Ax_1 + Ax_2 = A(x_1 + x_2) = 2b$$

$$A\left(\frac{x_1 + x_2}{2}\right) = b$$

so if $x_1 = (6,2,2,)$ and $x_2 = (2,4,8)$ then $x_3 = \frac{x_1+x_2}{2} = (4,3,5)$ is also a solution.

c) $f(x), g(x)$ are strictly increasing and strictly concave functions. We can define f and g as follows:

$$f(x) = g(x) = x^{\frac{2}{3}}$$

$$f'(x) = g'(x) = \frac{2}{3}x^{-\frac{1}{3}} > 0 \; for \; x > 0$$

$$f''(x) = g''(x) = -\frac{2}{9}x^{-\frac{4}{3}} < 0 \; for \; x > 0$$

Then the product function h:

$$h(x) = f(x)g(x) = x^{\frac{4}{3}}$$

$$h'(x) = \frac{4}{3}x^{\frac{1}{3}} > 0 \; for \; x > 0$$

$$h''(x) = \frac{4}{9}x^{-\frac{2}{3}} > 0 \; for \; x > 0$$

so h is a strictly convex function and the statement is not true.

d) We have the following linear system:

$$\begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c \\ c \\ c \end{bmatrix}$$

We can derive the determinant of the coefficient matrix as follows:

$$\det\left(\begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix}\right) = a(a^2 - b^2) - b(ba - b^2) + b(b^2 - ab)$$

$$= a(a^2 - b^2) - 2b^2(a - b) = (a - b)(a^2 + ab - 2b^2) = (a - b)^2(a + 2b)$$

which is always positive if $a, b > 0$, $a \neq b$, so the system has a unique solution.

We use Cramers rule to determine x_1:

$$x_1 = \frac{\det\left(\begin{bmatrix} c & b & b \\ c & a & b \\ c & b & a \end{bmatrix}\right)}{\det\left(\begin{bmatrix} a & b & b \\ b & a & b \\ b & b & a \end{bmatrix}\right)} = \frac{c(a-b)^2}{(a-b)^2(a+2b)} = \frac{c}{a+2b} > 0$$

Question 2:

$$f_1 = x_1\big(a - b(x_1 + x_2)\big) - c_1 x_1^2$$

$$f_2 = x_2\big(a - b(x_1 + x_2)\big) - c_2 x_2^2$$

a) The first order conditions are as follows:

$$\frac{\partial f_1}{\partial x_1} = a - b(x_1 + x_2) - bx_1 - 2c_1 x_1 = a - 2(b + c_1)x_1 - bx_2$$

$$\frac{\partial f_2}{\partial x_2} = a - b(x_1 + x_2) - bx_2 - 2c_2 x_2 = a - 2(b + c_2)x_2 - bx_1$$

b,c)

$$a - 2(b + c_1)x_1 - bx_2 = 0$$

$$a - 2(b + c_2)x_2 - bx_1 = 0$$

Equivalently

$$2(b + c_1)x_1 + bx_2 = a$$

$$bx_1 + 2(b + c_2)x_2 = a$$

We use the Cramers rule to solve the system of equations:

$$x_1 = \frac{\det\left(\begin{bmatrix} a & b \\ a & 2(b+c_2) \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 2(b+c_1) & b \\ b & 2(b+c_2) \end{bmatrix}\right)} = \frac{a(b+2c_2)}{4(b+c_1)(b+c_2)-b^2}$$

$$x_2 = \frac{\det\left(\begin{bmatrix} 2(b+c_1) & a \\ b & a \end{bmatrix}\right)}{\det\left(\begin{bmatrix} 2(b+c_1) & b \\ b & 2(b+c_2) \end{bmatrix}\right)} = \frac{a(b+2c_1)}{4(b+c_1)(b+c_2)-b^2}$$

d)

As it is mentioned in the question, it is not possible to explicitly derive the solutions (x_1, x_2) in here, so we use implicit function theorem to obtain the derivative, $\frac{\partial x}{\partial c}$, to illustrate the changes in endogenous variables (x) when we change the exogenous ones (c). Since $x_1 = x_2$, $c_1 = c_2$, we have:

$$f(x) = xp(2x) - cx^2$$

$$g(x) = \frac{df}{dx} = p(2x) + 2xp'(2x) - 2cx = 0$$

$$\frac{\partial x}{\partial c} = \frac{\frac{\partial g}{\partial c}}{\frac{\partial g}{\partial x}} = \frac{-2x}{4p'(2x) + 4xp''(2x) - 2c} = \frac{-x}{2p'(2x) + 2xp''(2x) - c}$$

Question 3:

$$\max_{x,y} \alpha \ln(x) + \beta \ln(y)$$

$$st. p_x x + p_y y \le w$$

$$x, y > 0$$

a)

$$L = \alpha \ln(x) + \beta \ln(y) - \lambda(p_x x + p_y y - w)$$

first order conditions are:

$$\frac{\partial L}{\partial x} = \frac{\alpha}{x} - \lambda p_x = 0$$

$$\frac{\partial L}{\partial y} = \frac{\beta}{y} - \lambda p_y = 0$$

$$\lambda(p_x x + p_y y - w) = 0$$

b) Starting with the assumption that $\lambda = 0$, we have:

$$\frac{\alpha}{x} = 0 \text{ and } \frac{\beta}{y} = 0$$

which are not possible since x and y cannot take infinite values, so $\lambda \neq 0$ and the budget constraint is binding and

$$p_x x + p_y y = w$$

c)

$$\max_{x,y,z} \alpha \ln(x+z) + \beta \ln(y+z)$$
$$st.\, p_x x + p_y y + p_z z \le w$$
$$x, y, z \ge 0$$

We now form the lagrangian and the first order conditions:

$$L = \alpha \ln(x+z) + \beta \ln(y+z) - \lambda_1\left(xp_x + yp_y + zp_z - w\right) + \lambda_2 x + \lambda_3 y + \lambda_4 z$$

$$\frac{\partial L}{\partial x} = \frac{\alpha}{x+z} - \lambda_1 p_x + \lambda_2 = 0$$

$$\frac{\partial L}{\partial y} = \frac{\beta}{y+z} - \lambda_1 p_y + \lambda_3 = 0$$

$$\frac{\partial L}{\partial z} = \frac{\alpha}{x+z} + \frac{\beta}{y+z} - \lambda_1 p_z + \lambda_4 = 0$$

$$\lambda_1\left(xp_x + yp_y + zp_z - w\right) = 0$$

$$\lambda_2 x = 0$$

$$\lambda_3 y = 0$$

$$\lambda_4 z = 0$$

If we set $z = 0$, the first two conditions will be as follows:

$$\frac{\partial L}{\partial x} = \frac{\alpha}{x} - \lambda_1 p_x + \lambda_2 = 0$$

$$\frac{\partial L}{\partial y} = \frac{\beta}{y} - \lambda_1 p_y + \lambda_3 = 0$$

It seems obvious that $x, y > 0$, otherwise the denominator will be zero.

d) Assuming that $x, y, z > 0$, we have the following conditions:

$$\frac{\partial L}{\partial x} = \frac{\alpha}{x+z} - \lambda_1 p_x = 0$$
$$\frac{\partial L}{\partial y} = \frac{\beta}{y+z} - \lambda_1 p_y = 0$$
$$\frac{\partial L}{\partial z} = \frac{\alpha}{x+z} + \frac{\beta}{y+z} - \lambda_1 p_z = 0$$
$$\lambda_1\left(xp_x + yp_y + zp_z - w\right) = 0$$

We can easily prove that budget constraint is binding since by setting $\lambda_1 = 0$, we have

$$\frac{\alpha}{x+z} = \frac{\beta}{y+z} = 0$$

which is not possible in here, so $xp_x + yp_y + zp_z = w$.

From the first two equations we have:

$$\frac{\alpha}{x+z} = \lambda_1 p_x$$
$$\frac{\beta}{y+z} = \lambda_1 p_y$$

Putting them inside the third one, we have

$$\lambda_1 p_x + \lambda_1 p_y = \lambda_1 p_z \; and \; \lambda_1 \neq 0$$

$$p_x + p_y = p_z$$

We derive this result without having any additional assumption, so we can use this in our FOC conditions. The resulting system of equations are as follows:

$$\frac{\alpha}{x+z} - \lambda_1 p_x = 0$$
$$\frac{\beta}{y+z} - \lambda_1 p_y = 0$$
$$\frac{\alpha}{x+z} + \frac{\beta}{y+z} - \lambda_1 p_x - \lambda_1 p_y = 0$$
$$\lambda_1 \left(x p_x + y p_y + z p_z - w \right) = 0$$

It is obvious that the third equation is just the sum of the first two ones. As the result we have three identifying equations and four variables, so this system has infinite number of solutions.

e) The resulting system of equations are as follows:

$$\frac{\alpha}{x} - \lambda_1 p_x = 0$$
$$\frac{\beta}{y} - \lambda_1 p_y = 0$$
$$\frac{\alpha}{x} + \frac{\beta}{y} - \lambda_1 p_z + \lambda_4 = 0$$
$$\lambda_1 \left(x p_x + y p_y - w \right) = 0$$

The budget constraint is binding so $\lambda_1 \neq 0$. The easiest way to solve this system is to derive x and y as a function of λ_1 from the first two equations. Then we can use the budget constraint to derive $\lambda_1$and finally we have x and y.

$$x = \frac{\alpha}{\lambda_1 p_x} \; , y = \frac{\beta}{\lambda_1 p_y}$$

using budget constraint:

$$\frac{\alpha}{\lambda_1} + \frac{\beta}{\lambda_1} = \frac{\alpha + \beta}{\lambda_1} = w \rightarrow \lambda_1 = \frac{\alpha + \beta}{w}$$

and

$$x = \frac{\alpha w}{p_x(\alpha + \beta)}$$

$$y = \frac{\beta w}{p_y(\alpha + \beta)}$$

Question 4:

a) We have

$$x_1 = wp_1^{-0.7}, x_2 = wp_2^{-0.3}$$

Using the utility maximization condition, we have

$$\frac{MU_x}{MU_y} = \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = \frac{p_x}{p_y}$$

Now we can use the demand functions to write prices as functions of demand values:

$$\frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = \frac{p_x}{p_y} = \frac{(\frac{w}{x_1})^{\frac{10}{7}}}{(\frac{w}{x_2})^{\frac{10}{3}}}$$

We need to find only one specific utility function, so we can easily assume:

$$\frac{\partial u}{\partial x_1} = (\frac{w}{x_1})^{\frac{10}{7}}$$

$$\frac{\partial u}{\partial x_2} = (\frac{w}{x_2})^{\frac{10}{3}}$$

We should integrate over x_1, x_2 to derive the utility function:

$$u_1 = \frac{w^{\frac{10}{7}} x_1^{-\frac{3}{7}}}{-\frac{3}{7}}$$

$$u_2 = \frac{w^{\frac{10}{3}} x_2^{-\frac{7}{3}}}{-\frac{7}{3}}$$

Finally, we can write

$$u = u_1 + u_2$$

b) The cost function of the firm is equal to

$$c(k, l) = rk + wl$$

where r and w are the prices for capitals and labor (wage) respectively.

We can easily derive the conditional labor demand by taking a partial derivative of the cost function with respect to wage(result of the envelope theorem), so

$$l(r, w, q) = \frac{\partial c}{\partial w} = 3q \quad if\ 2r > 3w$$

c) We have

$$x_{t+1} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} x_t = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}^{t+1} x_0$$

If we can write the matrix A in the form of

$$A = PDP^{-1}$$

we can easily calculate x_t for large values of t. To do this, we start with the eigenvalues and eigenvectors.

The eigenvalues of the matrix are as follows:

$$\lambda_1 = 3\,, \lambda_2 = \frac{-1-\sqrt{5}}{2} \cong -1.6\,, \lambda_3 = \frac{-1+\sqrt{5}}{2} \cong 0.6$$

and the eigenvectors are as follows:

for $\lambda_1 = 3$:

$$v_1 = \begin{bmatrix} 1.25 \\ 0.5 \\ 1 \end{bmatrix}$$

for $\lambda_2 = -1.6$:

$$v_2 = \begin{bmatrix} -0.6 \\ -0.4 \\ 1 \end{bmatrix}$$

for $\lambda_3 = 0.6$

$$v_3 = \begin{bmatrix} 1.6 \\ -2.6 \\ 1 \end{bmatrix}$$

Now we can form matrices P and D.

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1.6 & 0 \\ 0 & 0 & 0.6 \end{bmatrix}$$

and

$$P = \begin{bmatrix} 1.25 & -0.6 & 1.6 \\ 0.5 & -0.4 & -2.6 \\ 1 & 1 & 1 \end{bmatrix}$$

$$x_t = \begin{bmatrix} 1.25 & -0.6 & 1.6 \\ 0.5 & -0.4 & -2.6 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3^t & 0 & 0 \\ 0 & (-1.6)^t & 0 \\ 0 & 0 & 0.6^t \end{bmatrix} \begin{bmatrix} 1.25 & -0.6 & 1.6 \\ 0.5 & -0.4 & -2.6 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

We know that $\lambda_1 = 3 > 1$, so if $c_1 = v^{-1}x_0$ is not zero then the solution will be unstable. We can easily calculate vector c:

$$c = \begin{bmatrix} 0.36 \\ -0.51 \\ 0.14 \end{bmatrix}$$

Note that we only needed to calculate c_1. As the result we have:

$$x_t = v_1(3)^t.(0.36)$$

so the solution is unstable.