Mathematics for Economists
ECON-C1000
Spring 2022
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Problem Set 5, Due May 25, 2022: Retake Exam from 2021

You have four hours to complete exam. Please answer all four questions, they have equal weight. It is advisable to include an explanation of what you are trying to do.

1. Determine if the following statements are true or false. For true statements, find a proof and for false statements provide a counterexample or a reason why you consider the statement to be false. Explain the terms in italics.

   (a) If a matrix $\boldsymbol{A}$ is *non-singular*, then zero is one of its *eigenvalues*.

   (b) If vectors $(6, 2, 2)$ and $(2, 4, 8)$ solve the linear system

   $$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},$$

   then vector $(4, 3, 5)$ also solves the same system.

   (c) Suppose that $f(x)$ and $g(x)$ are defined for all $x > 0$. Suppose also that both f and g are positive (i.e. $f(x) > 0$ and $g(x) > 0$ for all $x > 0$), *strictly increasing* and *strictly concave* functions. Then the product function h given by $h(x) = f(x)g(x)$ is also strictly increasing and strictly concave. (Hint: you may assume if you want that f and g are twice differentiable).

   (d) Consider the following linear system:

   $$\begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} c \\ c \\ c \end{pmatrix},$$

   where a, b, c are all positive real numbers. If $a \neq b$, then the system has a unique solution and the solution is positive.

2. Consider the pair of functions for $i \in \{1, 2\}$:

   $$f_i(x_1, x_2) = x_i(a - b\sum_{j=1}^{2} x_j) - c_i x_i^2.$$

(a) Find the equations for the first-order conditions:

$$\frac{\partial f_1(\hat{x}_1, \hat{x}_2)}{\partial x_1} = 0,$$
$$\frac{\partial f_2(\hat{x}_1, \hat{x}_2)}{\partial x_2} = 0.$$

(b) For which parameters a, b, c_1, c_2 are the solutions $(\hat{x}_1, \hat{x}_2)$ non-negative?

(c) How does the solution $\hat{x}_1$ depend on c_1 and c_2?

(d) For a more general functional form:

$$f_i(x_1, x_2) = x_i p(x_1 + x_2) - c_i x_i^2,$$

where $p(\cdot)$ is a strictly concave and strictly decreasing function, it is not possible in general to solve explicitly the system:

$$\frac{\partial f_1(\hat{x}_1, \hat{x}_2)}{\partial x_1} = 0,$$
$$\frac{\partial f_2(\hat{x}_1, \hat{x}_2)}{\partial x_2} = 0.$$

Suppose that you are told that at $c_1 = c_2 = c$ the system has a symmetric solution $\hat{x}_1 = \hat{x}_2$. Explain how you would determine the comparative statics of the solution when c_1 increases.

3. Consider the consumer optimization problem

$$\max_{x,y} \alpha \ln(x) + \beta \ln(y)$$

subject to

$$p_x x + p_y y \leq w.$$
$$x, y > 0.$$

(a) Write the Lagrangean for the problem noting that the non-negativity constraint cannot bind since $x, y > 0$ and write the Kuhn-Tucker conditions for the problem.

(b) Argue that the budget constraint is binding and solve the problem.

(c) Add a third good z to the model so that the new problem becomes:

$$\max_{x,y,z} \alpha \ln(x + z) + \beta \ln(y + z)$$

subject to

$$p_x x + p_y y + p_z z \leq w,$$
$$x, y, z \geq 0,$$

where you can define $\ln(0) = -\infty$. Argue that if $z = 0$ at optimum, then $x > 0$ and $y > 0$.

(d) Is it possible to have at optimum $x > 0, y > 0, z > 0$?

(e) When is $z = 0$ at optimum?

4. Answer the following short questions:

(a) Can you find a utility function $u(x_1, x_2)$ resulting in the following demand functions for the two goods x_1, x_2:

$$x_1(p_1, p_2, w) = wp_1^{-0.7}, \text{ and } x_2(p_1, p_2, w) = wp_2^{-0.3},$$

where $w > 0$ is the income and $p_1, p_2 > 0$ are the prices of the goods.

(b) What is the cost function of a firm that buys labor input at price $w > 0$ and capital at price r? Suppose that the cost function is given by:

$$c(q, r, w) = \begin{cases} 2qr & \text{if} \quad 2r < 3w, \\ 3qw & \text{if} \quad 2r > 3w, \end{cases}$$

where q is the quantity produced. What is the conditional labor demand of the firm if $2r > 3w$?

(c) Examine the long-run behavior ($\lim_{t\to\infty} \boldsymbol{x}_t$) of the solution of the following linear system with initial condition $\boldsymbol{x}_0 = (1, 0, 0)$:

$$\boldsymbol{x}_{t+1} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix} \boldsymbol{x}_t.$$

(Hint: for one of the eigenvalues, consider the column sums of the matrix).