Mathematics for Economists
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Juuso Välimäki

# Maximum value function and envelope theorem

## The value function for utility maximization

Consider an unconstrained maximization problem of a function of a single real variable $x$, where the objective function depends on a parameter $\alpha \in \mathbb{R}$.

$$\max_{x \in \mathbb{R}} f(x, \alpha).$$

Let $x(\alpha)$ be the solution to this problem. Consider the maximum value of the objective function that is achievable at the exogenous variable (or parameter) $\hat{\alpha}$, i.e. $f(x(\hat{\alpha}), \hat{\alpha})$.

We call this new function the *value function* of the problem and denote

$$V(\alpha) := f(x(\alpha), \alpha).$$

At the (unconstrained) optimum $x(\hat{\alpha})$, by the first-order condition:

$$\frac{\partial f(x(\hat{\alpha}), \hat{\alpha})}{\partial x} = 0.$$

Suppose that $f$ is twice continuously differentiable and that the second order condition is satisfied so that

$$\frac{\partial^2 f(x(\hat{\alpha}), \hat{\alpha})}{\partial x^2} < 0.$$

Then we can use implicit function theorem to see that $x(\alpha)$ satisfying the first-order condition $\frac{\partial f(x(\hat{\alpha}), \hat{\alpha})}{\partial x} = 0$ exists in some neighborhood of $\hat{\alpha}$. We can compute via the chain rule:

$$V'(\hat{\alpha}) = \frac{\partial f(x(\hat{\alpha}), \hat{\alpha})}{\partial x} x'(\hat{\alpha}) + \frac{\partial f(x(\hat{\alpha}), \hat{\alpha})}{\partial \alpha}.$$

Since $\frac{\partial f(x(\hat{\alpha}), \hat{\alpha})}{\partial x} = 0$, we get

$$V'(\alpha) = \frac{\partial f(x(\alpha), \alpha)}{\partial \alpha}.$$

This observation is called the envelope theorem. In words, it states that when a parameter changes, the maximum value of the problem changes only through the direct effects on the objective function. The indirect effects on the value vanish because of the first-order condition on $x$.

In the more general case, where $\boldsymbol{x} \in \mathbb{R}^n$, the message is exactly the same. The first order-condition is now:

$$\frac{\partial f(\boldsymbol{x}(\hat{\alpha}), \hat{\alpha})}{\partial x_i} = 0 \text{ for all } i \in \{1, ..., n\}.$$

Assuming the conditions for implicit function theorem, we have by chain rule:

$$V'(\hat{\alpha}) = \sum_{i=1}^{n} \frac{\partial f(\boldsymbol{x}(\hat{\alpha}), \hat{\alpha})}{\partial x_i} x_i'(\hat{\alpha}) + \frac{\partial f(\boldsymbol{x}(\hat{\alpha}), \hat{\alpha})}{\partial \alpha}.$$

Again, the first term vanishes by first-order condition and we are left with

$$V'(\hat{\alpha}) = \frac{\partial f(\boldsymbol{x}(\hat{\alpha}), \hat{\alpha})}{\partial \alpha}.$$

The situation is slightly different with constrained optimization problems. Suppose that we have an equality constrained parametric maximization problem for $\boldsymbol{x} \in \mathbb{R}^n$:

$$\max_{\boldsymbol{x}} f(\boldsymbol{x}, \alpha)$$
$$\text{subject to } g(\boldsymbol{x}, \alpha) = 0.$$

Please note that the problem may have many parameters so that $\alpha$ is a vector, but the analysis here is with respect to a single component of the parameter vector.

Let $\boldsymbol{x}(\alpha)$ denote the optimal solution to the problem and assume again that he conditions for the implicit function theorem around the solution are satisfied as before. The value function is still defined as before:

$$V(\alpha) = f(\boldsymbol{x}(\alpha), \alpha).$$

We begin the analysis by forming the Lagrangean:

$$\mathcal{L}(\boldsymbol{x}, \mu; \alpha) = f(\boldsymbol{x}, \alpha) - \mu g(\boldsymbol{x}, \alpha).$$

Envelope theorem relates the derivative of the value function with respect to the parameter to the partial derivatives of the Lagrangean.

**Theorem 1** (Envelope theorem)**.** In an optimization problem subject to an equality constraint, we have:

$$V'(\alpha) = \frac{\partial \mathcal{L}(\boldsymbol{x}, \mu; \alpha)}{\partial \alpha}.$$

*Proof.*

$$V'(\hat{\alpha}) = \sum_{i=1}^{n} \frac{\partial f(\boldsymbol{x}(\hat{\alpha}), \hat{\alpha})}{\partial x_i} x_i'(\hat{\alpha}) + \frac{\partial f(\boldsymbol{x}(\hat{\alpha}), \hat{\alpha})}{\partial \alpha}.$$

First-order conditions for optimum imply that:

$$\frac{\partial f(\boldsymbol{x}(\hat{\alpha}), \hat{\alpha})}{\partial x_i} = \mu \frac{\partial g(\boldsymbol{x}(\hat{\alpha}), \hat{\alpha})}{\partial x_i}.$$

Since the $g(\boldsymbol{x}(\alpha), \alpha) = 0$ holds for all $\alpha$ near $\hat{\alpha}$, we have

$$\sum_{i=1}^{n} \frac{\partial g(\boldsymbol{x}(\hat{\alpha}), \hat{\alpha})}{\partial x_i} x_i'(\hat{\alpha}) = -\frac{\partial g(\boldsymbol{x}(\hat{\alpha}), \hat{\alpha})}{\partial \alpha}.$$

Combining these gives:

$$V'(\hat{\alpha}) = \frac{\partial f(\boldsymbol{x}(\hat{\alpha}), \hat{\alpha})}{\partial \alpha} - \mu \frac{\partial g(\boldsymbol{x}(\hat{\alpha}), \hat{\alpha})}{\partial \alpha}.$$

□

## Indirect utility function

The envelope theorem gives us a nice way of understanding the Lagrange multipliers in utility maximization problems. The Lagrangeam for the UMP with a single binding equality constraint is:

$$\mathcal{L}(\boldsymbol{x}, \lambda) = u(\boldsymbol{x}) - \mu \left[ \sum_{i=1}^{n} p_i x_i - w \right].$$

The maximum value function

$$v(\boldsymbol{p}, w) = \max u(\boldsymbol{x}) \text{ subject to } \boldsymbol{p} \cdot \boldsymbol{x} = w,$$

is called the indirect utility function. It computes the optimal utility level for all combinations of prices $\boldsymbol{p} \in \mathbb{R}^n_{++}$ and income $w > 0$.

Envelope theorem tells us that:

$$\frac{\partial v(\boldsymbol{p}, w)}{\partial w} = \mu.$$

This means that if your income is increased by one unit, your maximal utility increases the amount given by the multiplier. By reducing income $dw$ you lose $\mu dw$ of utility and this is why the multiplier is sometimes called the shadow price of income. It evaluates the utility consequences from relaxing or strengthening the constraint.

Envelope theorem also tells us that:

$$\frac{\partial v(\boldsymbol{p}, w)}{\partial p_i} = -\mu x_i.$$

Combining these two, we have Roy's identity:

$$x_i(\boldsymbol{p}, w) = -\frac{\frac{\partial v(\boldsymbol{p}, w)}{\partial p_i}}{\frac{\partial v(\boldsymbol{p}, w)}{\partial w}}.$$

In other words, if you have an indirect utility function, you can compute the demand function by simple partial differentiation. In later courses, you will learn what properties on $v(\boldsymbol{p}, w)$ guarantee that it is the indirect utility function of some UMP for some $u(\boldsymbol{x})$.

## Expenditure minimization

Consider next the expenditure minimization problem (EMP) from Lecture 9.

$$\min_{\boldsymbol{h} \in X} \boldsymbol{p} \cdot \boldsymbol{h} = \sum_{i=1}^{n} p_i h_i,$$

subject to

$$u(\boldsymbol{h}) = \overline{u}.$$

Denote the solution to this problem by $\boldsymbol{h}(\boldsymbol{p},\overline{u})$. We call $h_i(\boldsymbol{p},\overline{u})$ the Hicksian or compensated demand for good $i$. The (minimum) value function of this problem,

$$e(\boldsymbol{p},\overline{u}) = \sum_{i=1}^{n} p_i h_i(\boldsymbol{p},\overline{u}),$$

is called the expenditure function.

The objective function is linear in $\boldsymbol{p}$ and hence by the results in Lecture 6, we know that $e(\boldsymbol{p},\overline{u})$ is concave in $\boldsymbol{p}$. Therefore the Hessian matrix of $e(\boldsymbol{p},\overline{u})$ is negative semidefinite.

We turn next to the The Lagrangean function for the case where we can ignore the inequality constraints:

$$\mathcal{L}(\boldsymbol{x},\mu) = \sum_{i=1}^{n} p_i h_i - \mu(\overline{u} - u(\boldsymbol{h})).$$

The envelope theorem tells us that:

$$\frac{\partial e(\boldsymbol{p},\overline{u})}{\partial p_i} = h_i(\boldsymbol{p},\overline{u}).$$

The partial derivatives of $h_i(\boldsymbol{p},\overline{u})$ with respect to $p_j$ are the elements of the Hessian matrix of $e(\boldsymbol{p},\overline{u})$.

## Connecting UMP and EMP

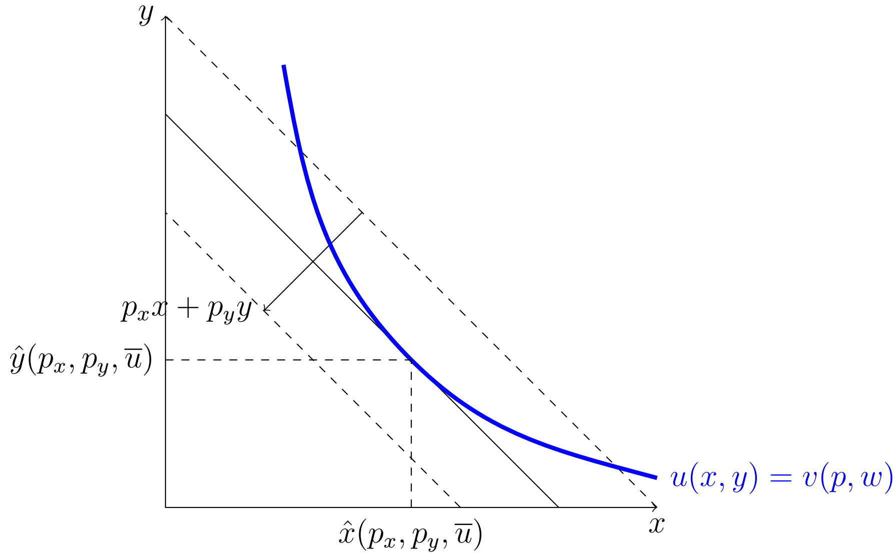
The main reason for considering the expenditure minimization problem is that it gives us a nice tool for understanding the solution to the utility maximization problem. Hold prices $\hat{\boldsymbol{p}}$ fixed for a moment and ask how high utility you can achieve with income $w$. The answer is given by the indirect utility function $v(\hat{\boldsymbol{p}},w)$.

Ask next what is the minimum expenditure that you must use to achieve utility $v(\hat{\boldsymbol{p}},w)$. The following figures should convince you that for all $\hat{\boldsymbol{p}}$,

$$e(\hat{\boldsymbol{p}}, v(\hat{\boldsymbol{p}},w)) = w.$$

Similarly, suppose that it costs you $e(\hat{\boldsymbol{p}},\overline{u})$ to reach utility $\overline{u}$. If your budget is $e(\hat{\boldsymbol{p}},\overline{u})$, then the maximal utility that you can reach is for all $\hat{p}$,

Figure 1: Expenditure minimization for $\overline{u} = v(\boldsymbol{p}, w)$



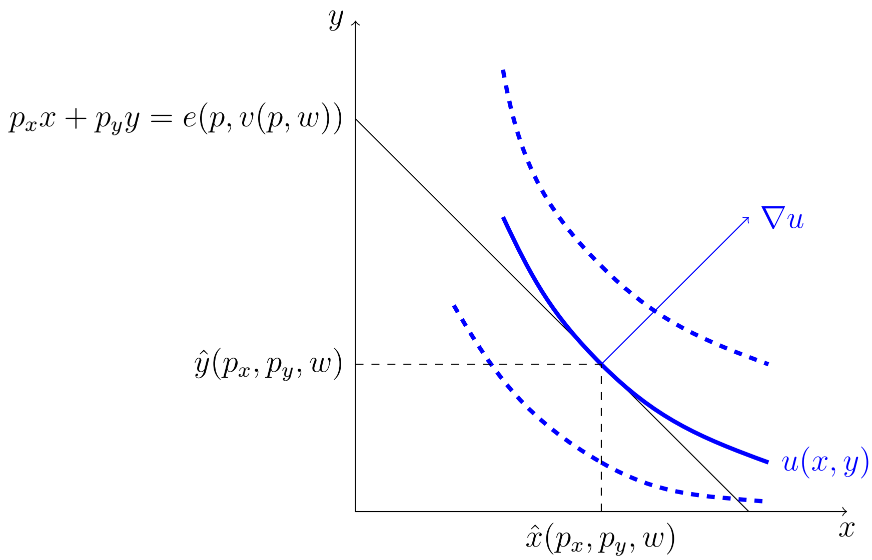

$$\overline{u} = v(\hat{\boldsymbol{p}}, e(\hat{\boldsymbol{p}}, \overline{u})).$$

This argument (or alternatively from the K-T constraints of the two problems), shows that for all $\boldsymbol{p}$, $\overline{u} = v(\boldsymbol{p}, e(\boldsymbol{p}, \overline{u}))$ and $e(\boldsymbol{p}, v(\boldsymbol{p}, w)) = w$ In other words, the solutions to expenditure minimization and UMP coincide for all $p$:

$$h_i(\boldsymbol{p}, \overline{u}) = x_i(\boldsymbol{p}, e(\boldsymbol{p}, \overline{u})) \text{ for all } i,$$

$$h_i(\boldsymbol{p}, v(\boldsymbol{p}, w)) = x_i(\boldsymbol{p}, w) \text{ for all } i.$$

Figure 2: UMP for $w = e(\boldsymbol{p}, v(\boldsymbol{p}, w))$




Differentiate the first of these identities with respect to $p_j$ to get:

$$\begin{aligned}
\frac{\partial h_i(\boldsymbol{p}, \overline{u})}{\partial p_j} &= \frac{\partial x_i(\boldsymbol{p}, w)}{\partial p_j} + \frac{\partial x_i(\boldsymbol{p}, w)}{\partial w}\frac{\partial e(\boldsymbol{p}, \overline{u})}{\partial p_j} \\
&= \frac{\partial x_i(\boldsymbol{p}, w)}{\partial p_j} + \frac{\partial x_i(\boldsymbol{p}, w)}{\partial w} h_j(\boldsymbol{p}, \overline{u}) \\
&= \frac{\partial x_i(\boldsymbol{p}, w)}{\partial p_j} + \frac{\partial x_i(\boldsymbol{p}, w)}{\partial w} x_j(p, e(\boldsymbol{p}, \overline{u})) \\
&= \frac{\partial x_i(\boldsymbol{p}, w)}{\partial p_j} + \frac{\partial x_i(\boldsymbol{p}, w)}{\partial w} x_j(\boldsymbol{p}, w).
\end{aligned}$$

This is the famous Slutsky equation for income and substitution effects. It relates the changes in the Marshallian (UMP) demands to the Hicksian (EMP) demands. Since the Marshallian demands depend on prices and income, they are in principle observable from demand data. The Hicksian demands depend on the utility level and hence they cannot be directly observed. Nevertheless, we know from the Hessian of the expenditure function that e.g. the Hicksian demand is downward sloping in own demand. With Slutsky equation, we can translate this knowledge to the Marshallian demands where the results are very hard to obtain directly.

The observable change in Marshallian $\frac{\partial x_i(\boldsymbol{p},w)}{\partial p_j}$ demands can be decomposed into a substitution effect, i.e. the change in compensated demand $\frac{\partial h_i(\boldsymbol{p},\overline{u})}{\partial p_j}$ and the observable income effect $\frac{\partial x_i(\boldsymbol{p},w)}{\partial w}x_j(\boldsymbol{p},w)$.

$$\frac{\partial x_i(\boldsymbol{p},w)}{\partial p_j} = \frac{\partial h_i(\boldsymbol{p},\overline{u})}{\partial p_j} - \frac{\partial x_i(\boldsymbol{p},w)}{\partial w}x_j(p,w).$$

Since we know that the Hessian of $e(p,\overline{u})$ is negative definite, we know that its diagonal elements are non-positive. Hence the effect of increasing $p_i$ on $x_i$ is negative whenever the demand for $i$ is increasing in income (we say then that $i$ is a non-inferior good).

## Cost minimization

A firm chooses its inputs $k,l$ to minimize the cost of reaching a production target of $q$ at given input prices $r,w$. The production function is assumed to be a strictly increasing and quasiconcave function $f(k,l)$.

$$\min_{(k,l)\in\mathbb{R}^2_+} rk + wl$$

subject to

$$f(k,l) = q.$$

The value function of this problem is called the *cost function* of the firm and denoted by $c(r,w,q)$. We write:

$$c(r,w,q) = rk(r,w,q) + wl(r,w,q),$$

where $k(r,w,q), l(r,w,q)$ solve the cost minimization problem. These are called the conditional factor demands.

As in the case with expenditure minimization, we see that the cost function is concave in $r,w$ since it is the minimum of linear functions of $r,w$. Therefore the Hessian of the cost function is negative semidefinite. By envelope theorem, we have the result known as Shephard's lemma:

$$\frac{\partial c(r,w,q)}{\partial r} = k(r,w,q), \qquad \frac{\partial c(r,w,q)}{\partial w} = l(r,w,q).$$

Negative semi-definiteness of the Hessian of $c$ implies that (since the diagonal elements must be non-positive)

$$\frac{\partial k(r,w,q)}{\partial r} \leq 0, \frac{\partial l(r,w,q)}{\partial w} \leq 0.$$

In words, conditional factor demands are decreasing in own price (not surprisingly).

## Profit function of a competitive firm

We end this part of the course with the analysis of profit maximization for a price taking firm. There are two ways to think about this. Either minimize cost for each production level $q$ to get $c(r,w,q)$ and then choose $q$ optimally to maximize

$$pq - c(r,w,q),$$

where $p$ is the price of the output.

Alternatively, you can write directly:

$$\max_{k,l,q} pq - rk - wl$$

subject to

$$q = f(k,l).$$

An advantage of the second approach is that the problem is immediately seen to be linear in the input and output prices $p, r, w$. Let

$$(q(p,r,w), k(p,r,w), l(p,r,w)),$$

be the optimal output and input choices in the problem. The value function $\pi(p,r,w)$ is called the profit function of the firm.

Since $\pi$ is the maximum of linear functions in $p, r, w$, we get by Lecture 6 that $\pi$ is convex and hence its Hessian is positive semi-definite.

The envelope theorem gives us Hotelling's lemma:

$$\frac{\partial \pi(p,r,w)}{\partial p} = q(p,r,w), \quad \frac{\partial \pi(p,r,w)}{\partial r} = -k(p,r,w), \quad \frac{\partial \pi(p,r,w)}{\partial w} = -l(p,r,w).$$

Since $\pi$ is positive semi-definite, its diagonal elements are non-negative. This gives the 'Law of Supply' (supply increases in output price):

$$\frac{\partial q(p,r,w)}{\partial p} \geq 0,$$

and the 'Law of Factor Demands' (factor demand decrease in factor price):

$$\frac{\partial k(p,r,w)}{\partial r} \leq 0, \quad \frac{\partial l(p,r,w)}{\partial w} \leq 0.$$

As you can see, the theory of the competitive firm is easier than consumer theory since changes in prices do not change the constraint set (as with the budget set). You will see the firm's problem in some form in almost all branches of economics and in particular in Intermediate Microeconomics. Of course, in many industries firms are not competitive $\rightarrow$ Industrial organization.

On the other hand, firms do not make decisions but people do and people may have different objectives $\rightarrow$ Organizational economics, Contract theory.