Mathematics for Economists
ECON-C1000
Spring 2021
Juuso Välimäki
juuso.valimaki@aalto.fi

Problem Set 5: Practice Exam and two extra problems, Due May 27 at 10am

In the exam, you would have four hours to complete exam. I would ask you to answer the four first questions, they have equal weight. The last two questions are here to give you a chance to get more exercise with the material of the last week.

1. Consider the following systems of linear equations.

   (a) Explain (in at most four lines) what it means if an $n \times n$ matrix A has full rank. Determine c so that the matrix A below does **not** have full rank

   $$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \\ 0 & 6 & c \end{pmatrix}.$$

   (b) Consider the system of equations given by

   $$Ax = y,$$

   where A is an $n \times n$ matrix. Show that if the system has two distinct solutions x^1 and x^2 for some y, then A does not have full rank.

   (c) What are the eigenvalues of the following matrix? (Recall that the sum of eigenvalues is equal to the trace of the matrix and the product of the eigenvalues equals the determinant).

   $$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}.$$

   (d) Consider the following dynamical system

   $$x_{t+1} = x_t + 2y_t,$$
   $$y_{t+1} = -x_t + 4y_t$$

   for an arbitrary initial value (x_0, y_0). What happens to (x_t, y_t) as $t \to \infty$?

2. Consider the following optimization problem for $x, y > 0$

   $$\max_{x,y} 2y^2(1 - \ln x) - \alpha x^2 - \beta y.$$

   (a) Write the first order conditions for this problem.

   (b) Does the point $(\hat{x}, \hat{y}, \hat{\alpha}, \hat{\beta}) = (1, 1, 1, 4)$ solve the first order conditions? Can you apply the implicit function theorem to show that $(x(\alpha, \beta), y(\alpha, \beta))$ satisfying the first order conditions exists in some neighborhood of $(\hat{x}, \hat{y}, \hat{\alpha}, \hat{\beta}) = (1, 1, 1, 4)$?

(c) Is $(\hat{x}, \hat{y}) = (1, 1)$ a maximum or a minimum for the problem at these parameter values for α and β?

3. Solve the consumer optimization problem

$$\max_{x,y} \left(\frac{1}{x^2} + \frac{1}{y^2}\right)^{-\frac{1}{2}}$$

subject to

$$p_x x + p_y y \leq w.$$

$$x, y > 0.$$

(a) Write the Lagrangean for the problem noting that the non-negativity constraint cannot bind since $x, y > 0$ and write the Kuhn-Tucker conditions for the problem.

(b) Argue that the budget constraint is binding and solve the problem.

(c) Find the partial derivatives for the optimal demand of good x, i.e. the partial derivatives $\frac{\partial x}{\partial p_x}$, $\frac{\partial x}{\partial p_y}$ and $\frac{\partial x}{\partial w}$ of $x(p_x, p_y, w)$.

4. The cost minimization problem of a competitive firm is to

$$\min_{k,l} rk + wl$$

subject to

$$f(k, l) = q,$$

where $f(k, l)$ is the production function of the firm and $q > 0$ is the production target of the firm.

(a) Explain in no more than four lines what is the value function of an optimization problem.

(b) Explain in no more than four lines what is the content of the envelope theorem for constrained optimization problems.

(c) Suppose that the value function of this problem, i.e. the cost function of the firm is:

$$c(r, w, q) = \theta q r^{\alpha} l^{(1-\alpha)}$$

for $0 < \alpha < 1$ and some constant θ. Use the envelope theorem to derive the solutions $k(r, w, q), l(r, w, q)$ to the cost minimization problem that gives rise to this cost function.

(d) Determine the partial derivatives of $k(r, w, q)$.

5. **Exercise.** Consider the following system of difference equations:

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix}, \quad t = 0, 1, 2, \ldots$$

Find the general solution of this system and determine whether it is stable or not.

6. What are the eigenvalues of the following matrix? (Recall that the sum of eigenvalues is equal to the trace of the matrix and the product of the eigenvalues equals the determinant).

$$\begin{pmatrix} 0.4 & 0 & 0.8 \\ 0.6 & 0.4 & 0 \\ 0 & 0.6 & 0.2 \end{pmatrix}.$$

Consider the following dynamical system in $\mathbb{R}^3$ with initial value $x_0 = y_0 = z_0 = \frac{1}{3}$.

$$\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0.4 & 0 & 0.8 \\ 0.6 & 0.4 & 0 \\ 0 & 0.6 & 0.2 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}.$$

What happens to (x_t, y_t, z_t) as $t \to \infty$?