Problem set 5

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Question 1:

a) A square matrix n by n is full rank if the rows and columns are linearly independent from each other.
for the following matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \\ 0 & 6 & c \end{bmatrix}$$

we set the determinant equal to zero and then find the value of c.

$$det = -6(5-3*4) + c(0-2*4) = 42 - 8c = 0 \rightarrow c = \frac{21}{4}$$

b)

$$Ax = y$$

the above equation has two distinct solutions x_1, x_2, so $x_1 \neq x_2$, so we can have:

$$Ax_1 - Ax_2 = y - y = 0 \rightarrow A(x_1 - x_2) = 0$$

We have from previous studies that if we have

$$Ax = 0 \ \ and \ \ x \neq 0 \ \rightarrow \det(A) = 0$$

so in our case we have

$$A(x_1 - x_2) = 0 \ and \ x_1 \neq x_2 \ \rightarrow \det(A) = 0$$

and matrix A does not have full rank.

c) we have to derive the eigenvalues of the matrix:

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$$

we know that the sum of the eigenvalues is equal to $tr(A)$ and product of them is equal to $\det(A)$. Assuming x ,y the eigenvalues:

$$x + y = tr(A) = 5$$
$$x * y = \det(A) = 6$$

solving the system of equations, we have $x = 2$, $y = 3$.

d)

$$x_{t+1} = x_t + 2y_t$$

$$y_{t+1} = -x_t + 4y_t$$

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}^{t+1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$$

We want to write the A matrix in the form:

$$A = PDP^{-1}$$

Where D is diagonal matrix.

We know the eigenvalues of the matrix A from part c, so we can easily derive the eigenvectors:

For $\lambda_1 = 2$:

$$\begin{bmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix} \rightarrow v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

For $\lambda_1 = 3$:

$$\begin{bmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix} \rightarrow v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So:

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \; AND \;\; D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

So after all

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = PD^tP^{-1}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 2^{t+1} - 3^t & -2^{t+1} + 2 * 3^t \\ 2^t - 3^t & -2^t + 2 * 3^t \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

And since $\lambda_1, \lambda_2 > 1$, the steady state of the system is not stable.

Question 2:

$$\left(\hat{x}, \hat{y}, \hat{\alpha}, \hat{\beta}\right) = (1,1,-1,4)$$

a)

$$\frac{\partial f}{\partial x} = -2\frac{y^2}{x} - 2\alpha x$$

$$\frac{\partial f}{\partial y} = 4y(1 - \ln(x)) - \beta$$

b) at the point $\left(\hat{x}, \hat{y}, \hat{\alpha}, \hat{\beta}\right) = (1,1,-1,4)$ we have:

$$\frac{\partial f}{\partial x}(1,1,-1,4) = -2 + 2 = 0$$

$$\frac{\partial f}{\partial y} = 4 - 4 = 0$$

to use implicit function theorem, we should derive D_y, so:

$$D_y = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} \frac{2y^2}{x^2} - 2\alpha & -4\frac{y}{x} \\ -4\frac{y}{x} & 4(1 - \ln(x)) \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}$$

so

$$\det(D_y) = 0$$

so we do not have the necessary condition to use implicit function theorem around the point.

c) since the hessian matrix of f at the point (1,1) is not positive or negative definite so the function f does not have a local maximum or local minimum at the point.

Question 3:

$$\max_{x,y} \left(\frac{1}{x^2} + \frac{1}{y^2}\right)^{-\frac{1}{2}}$$

$$st. p_x x + p_y y \leq w$$

$$x, y > 0$$

a) we first form the Lagrangean of the problem:

$$L = \left(\frac{1}{x^2} + \frac{1}{y^2}\right)^{-\frac{1}{2}} - \mu_1(p_x x + p_y y - w) + \mu_2 x + \mu_3 y = 0$$

Now we can write the Kuhn Tucker conditions as follows:

$$\frac{dL}{dx} = \left(\frac{-1}{2}\right)\left(\frac{-2}{x^3}\right)\left(\frac{1}{x^2} + \frac{1}{y^2}\right)^{-\frac{3}{2}} - \mu_1 p_x + \mu_2 = 0$$

$$\frac{dL}{dy} = \left(\frac{-1}{2}\right)\left(\frac{-2}{y^3}\right)\left(\frac{1}{x^2} + \frac{1}{y^2}\right)^{-\frac{3}{2}} - \mu_1 p_y + \mu_3 = 0$$

$$\mu_1(p_x x + p_y y - w) = 0$$

$$\mu_2 x = 0$$

$$\mu_3 y = 0$$

since x and y are strictly positive we can easily conclude that $\mu_2 = \mu_3 = 0$

b)

Using the first equation, we have:

$$\mu_1 = \frac{\left(\frac{1}{x^2}+\frac{1}{y^2}\right)^{-\frac{3}{2}}}{x^3 p_x}$$

and since x and y are positive numbers, there is no way $\mu_1 = 0$, so the budget constraint is binding and we have: $p_x x + p_y y = w$

To solve the problem, we can use the first two equations to eliminate μ_1 and find x as a function of y.

$$\frac{\left(\frac{1}{x^2}+\frac{1}{y^2}\right)^{-\frac{3}{2}}}{x^3 p_x} = \frac{\left(\frac{1}{x^2}+\frac{1}{y^2}\right)^{-\frac{3}{2}}}{y^3 p_y} \rightarrow \frac{x^3}{y^3} = \frac{p_y}{p_x} \rightarrow \frac{x}{y} = (\frac{p_y}{p_x})^{\frac{1}{3}}$$

using the binding budget constraint, we will have:

$$y = \frac{w}{p_y^{\frac{1}{3}} p_x^{\frac{2}{3}} + p_y}$$

$$x = \frac{w}{p_x^{\frac{1}{3}} p_y^{\frac{2}{3}} + p_x}$$

c)we write the partial derivatives as follows:

$$\frac{dx}{dw} = \frac{1}{p_x^{\frac{1}{3}} p_y^{\frac{2}{3}} + p_x}$$

$$\frac{dx}{dp_x} = -\frac{w(\frac{1}{3} p_x^{-\frac{2}{3}} p_y^{\frac{2}{3}} + 1)}{(p_x^{\frac{1}{3}} p_y^{\frac{2}{3}} + p_x)^2}$$

$$\frac{dx}{dp_y} = -\frac{w(\frac{2}{3} p_x^{\frac{1}{3}} p_y^{-\frac{1}{3}})}{(p_x^{\frac{1}{3}} p_y^{\frac{2}{3}} + p_x)^2}$$

Question 4:

The cost minimization problem:

$$\min_{k,l} rk + wl$$

$$st. f(k,l) = q$$

a) In an optimization problem where

$$\max_{x \in R} f(x,a)$$

the value function is the maximum value of the objective function that is achievable on the exogenous variable $\hat{\alpha}$. we will show the value function as the function of the exogenous variables:

$$V(\alpha) = f(x(\alpha), \alpha)$$

b) In an optimization problem with constraints, we have:

$$V'(\alpha) = \frac{\partial L(x, \mu; \alpha)}{\partial \alpha}$$

where L is the lagrangean of the optimization problem. The intuition behind the envelope theorem is that if we want to derive the effect of the parameters (α) on the lagrangean, we only need to consider the direct effect, and all the indirect effects will be zero because:

$$\frac{\partial L}{\partial x} = 0$$

c) the cost function of the firm is:

$$c(r, w, q) = \theta q r^{\alpha} w^{1-\alpha}$$

using the envelope theorem we have: (q , r , w are exogenous variables)

$$\frac{\partial c}{\partial r} = k = \theta q \alpha r^{\alpha-1} w^{1-\alpha} \to k = \theta q \alpha r^{\alpha-1} w^{1-\alpha}$$

$$\frac{\partial c}{\partial w} = l = \theta(1-\alpha) q r^{\alpha} w^{-\alpha} \to l = \theta(1-\alpha) q r^{\alpha} w^{-\alpha}$$

d) partial derivatives of k:

$$\frac{dk}{dr} = \theta q \alpha(\alpha - 1) r^{\alpha-2} l^{1-\alpha}$$

$$\frac{dk}{dw} == \theta q \alpha(1-\alpha) r^{\alpha-1} w^{-\alpha}$$

$$\frac{dk}{dq} = \theta \alpha r^{\alpha-1} w^{1-\alpha}$$

Question 5:

our problem is:

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix}$$

We want to write the A matrix in the form:

$$A = PDP^{-1}$$

Where D is diagonal matrix.

The eigenvalues of the matrix A are $\lambda_1 = 4.79\ and\ \lambda_2 = 0.21$

so we can easily derive the eigenvectors:

For $\lambda_1 = 4.79$:

$$A - \lambda I = \begin{bmatrix} -4.79 & 1 \\ -1 & 0.21 \end{bmatrix} \rightarrow v_1 = \begin{bmatrix} 1 \\ 4.79 \end{bmatrix}$$

For $\lambda_1 = 0.21$:

$$A - \lambda I = \begin{bmatrix} -0.21 & 1 \\ -1 & 4.79 \end{bmatrix} \rightarrow v_1 = \begin{bmatrix} 1 \\ 0.21 \end{bmatrix}$$

So:

$$P = \begin{bmatrix} 1 & 1 \\ 4.79 & 0.21 \end{bmatrix}\ AND\ \ D = \begin{bmatrix} 4.79 & 0 \\ 0 & 0.21 \end{bmatrix}$$

So after all

$$\begin{bmatrix} x_t \\ y_t \end{bmatrix} = PD^tP^{-1}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4.79 & 0.21 \end{bmatrix}\begin{bmatrix} 4.79^t & 0 \\ 0 & 0.21^t \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 4.79 & 0.21 \end{bmatrix}^{-1}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

And since one of the eigenvalues is greater than one $\lambda_1 > 1$, the steady state of the system is not stable.

Question 6:

The transform matrix is:

$$A = \begin{bmatrix} 0.4 & 0 & 0.8 \\ 0.6 & 0.4 & 0 \\ 0 & 0.6 & 0.2 \end{bmatrix}$$

since matrix A is a stochastic matrix, one eigenvalue is equal to 1 (we calculated in the first problem set). We also know that the sum of the eigenvalues is equal to the tr(A) and product of the is equal to the det(A), so

$$\lambda_1 + \lambda_2 = 0$$

$$\lambda_1 . \lambda_2 = \det(A) = 0.32$$

to find the eigenvector for the $\lambda = 1$ we have:

$$A - \lambda I = \begin{bmatrix} 0 & -0.6 & 0.8 \\ 0.6 & -0.6 & 0 \\ 0 & 0.6 & -0.8 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & -0.6 & 0.8 \\ 0.6 & -0.6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow$$

$$0.6x_2 = 0.8x_3\ and\ x_1 = x_2$$

finally, the eigen vector is:

$$v = \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix}$$

The absolute value of the second and third eigenvalue is less than one so as t goes to infinity the effect of them is going to be zero, and x_t in the long run will converge to the normalized vector of the eigenvector corresponding to the eigenvalue 1, which is

$$v_1 = \begin{bmatrix} \frac{4}{11} \\ \frac{4}{11} \\ \frac{3}{11} \end{bmatrix}$$