Lecture 9:
Why donuts are so good …

Menu of today

- Why donuts are so good: Hairy ball theorem

- Drift *orbits*

- Neoclassical transport

- Noether's theorem

- Tokamak ABC

- More on 'anomalous' transport:

  - TF ripple,

  - microinstabilities

- Drifts due to non-stationary fields

- Plasma heating by adiabatic compression

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Depressing news from last week

Previously, in IPPFSA …

Analytical work led to findings that are **4 orders of magnitude** from reality:

- We derived expression for the diffusion coefficient D_{cl} and let the continuity equation give us the *confinement time*

We already identified a few possible causes for increased *transport* – not necessarily *diffusive* in nature.

But so far we ignored the effect of *geometry!*

- Recall lecture#2: in linear geometry (cylinder) we have a loss cone

$$\frac{v_{\parallel,0}^2}{v_0^2} \equiv \xi_0^2 > 1 - B_{min}/B_{max}$$

These are called *end losses* and they certainly add to diffusive losses.

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Loss cone and collisions
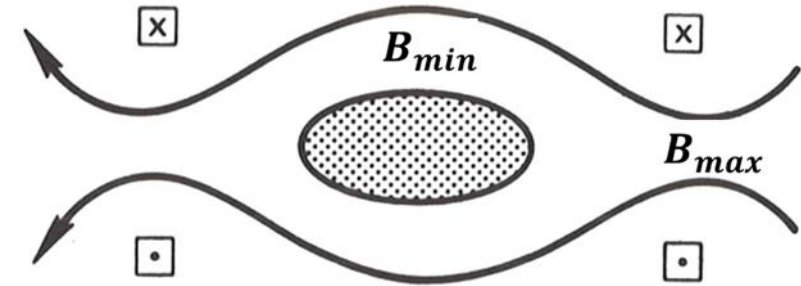
Features of the end losses:

- Losses via loss cone are along field lines, and, thus, almost immediate

  ➔ loss cone is 'empty' in no time

Recall the effect of Coulomb collisions:

- Even small-angle collisions change the direction of a particle

  ➔ a particle with $\xi_0^2 < 1 - B_{min}/B_{max}$ can be kicked to the loss cone

Here even like-particle collisions can feed the loss cone

➔ Eventually entire plasma depleted at rate determined by ν_{coll}

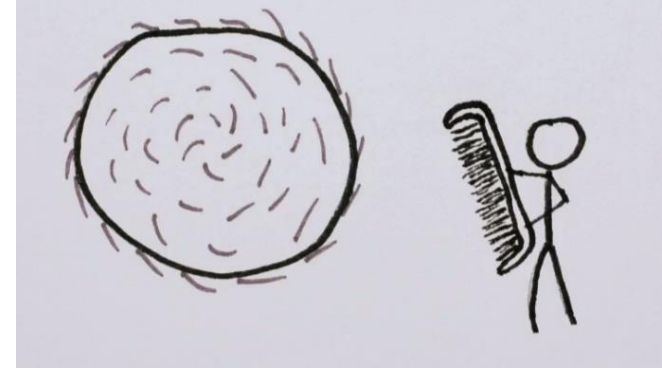Note: loss cone independent of particle species (*q,m*)

How to eliminate end losses?

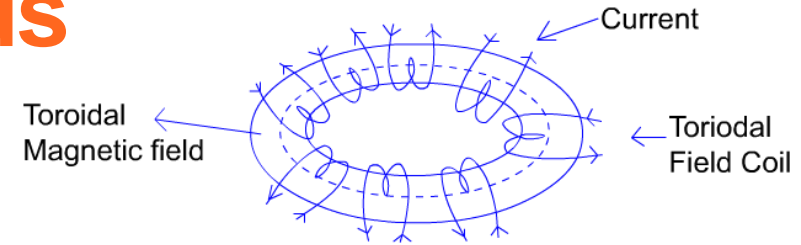This is equal to asking: ***"how to confine magnetic field lines"***?

The answer is given by a mathematical theorem in algebraic topology, also known as

The Hairy Ball theorem

https://www.youtube.com/watch?v=B4UGZEjG02s

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Magnetic field lines in a torus

When the axial field is turned into a toroidal one, it becomes non-uniform.

In the first approximation (geometrical considerations): $B = B_0 \frac{R_0}{R}$

(Previous "*Food for thought*"): $v_{\nabla B} = \pm \frac{1}{2} v_\perp r_L \frac{\boldsymbol{B} \times \nabla B}{B^2}$ (HW: for 1/R field)

➔ Electrons move upward, ions downward (or vice versa)

➔ A vertical electric field is established!

➔ ExB drift makes both species march horizontally out of the cage …

The need for *helical* field lines

The physics approach to field line problem:

* Having fixed the geometry as torus, the gradient drift is unavoidable
* The charge separation *can* be prevented:

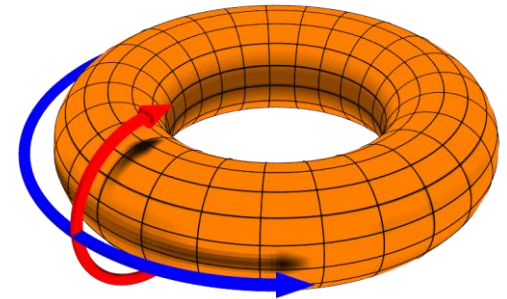 Short-circuit the system!

The electrons need a route to the ions ➔ a vertical B-field?

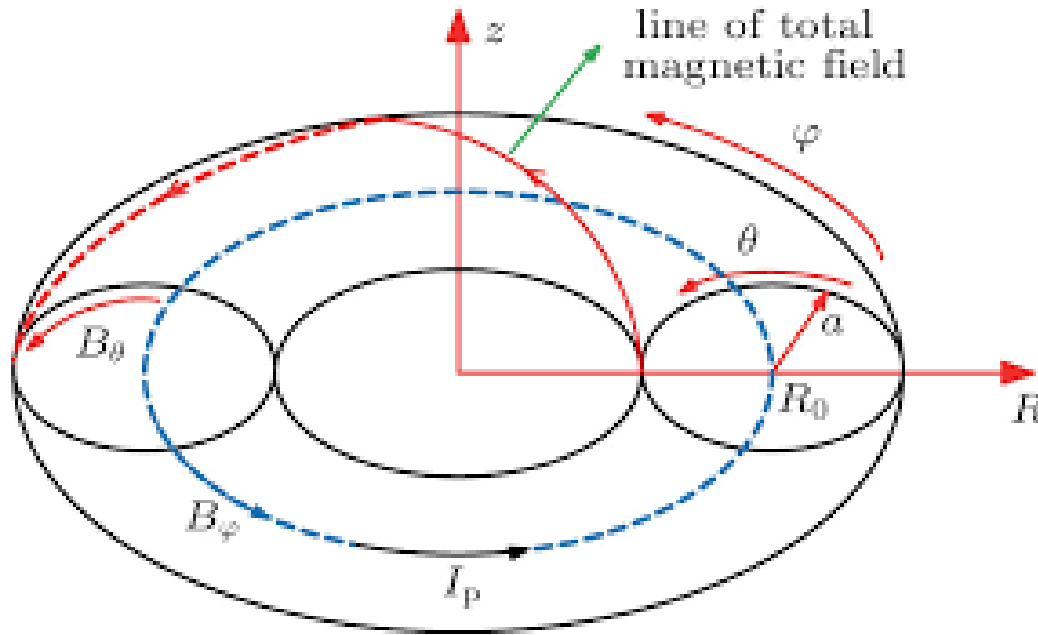No, that is not natural to toroidal geometry

➔ introduce a

  poloidal magnetic field

to supplement the *toroidal* field

Toroidal magnetic cage

Toroidal angle: φ
Poloidal angle: θ
Toroidal field: B_φ
Poloidal field: B_θ
Major radius: R
Minor radius: r (a)
Magnetic axis at $R = R_0$

There is no such thing as free lunch

Things look perfect now – or do they?

There is a price to pay for the more complicated geometry …

Life in toroidal geometry

Single particle in *non-uniform* B-field Part III: curved B

In a tokamak, the field lines have *curvature*, R_c

Geometry makes math complicated ➔ let's take a more intuitive approach:

A particle on a curved path experiences centrifugal force $\boldsymbol{F}_{cf} = \dfrac{mv_{\parallel}^2}{R_c}\boldsymbol{r} = mv_{\parallel}^2\dfrac{\boldsymbol{R_c}}{R_c^2}$

➔ *GC* drift called the **curvature drift**: $v_R = \dfrac{1}{q}\dfrac{\boldsymbol{F}_{cf}\times\boldsymbol{B}}{B^2} = \dfrac{mv_{\parallel}^2}{qB^2}\dfrac{\boldsymbol{R_c}\times\boldsymbol{B}}{R_c^2}$

But this is not all, folks …

When modifying our magnetic field on pen-and-paper, we have to make sure our new field is *physical* !

Curvature drift with a physical B field

A curved field cannot be uniform ➜ accompanying gradient drift.

Ampere' law in vacuum: $\nabla \times \boldsymbol{B} = 0$

Cylindrical coordinates (natural choice): $\nabla \times \boldsymbol{B} = (\nabla \times \boldsymbol{B})_z = \frac{1}{R}\frac{\partial}{\partial R}\left(RB_\varphi\right) = 0$

➜ $B_\varphi \propto \frac{1}{R}$ ➜ $B \propto \frac{1}{R_c}$ ➜ $\frac{\nabla B}{B} = -\frac{\boldsymbol{R}_c}{R_c^2}$ accompanying $\boldsymbol{v}_{\nabla B} = \frac{1}{2}\frac{m}{q}v_\perp^2\frac{\boldsymbol{R}_c \times \boldsymbol{B}}{R_c^2 B^2}$

➜ Total drift in curved B field: $\boldsymbol{v_R} + \boldsymbol{v}_{\nabla B} = \frac{m}{q}\left(v_\parallel^2 + \frac{1}{2}v_\perp^2\right)\frac{\boldsymbol{R}_c \times \boldsymbol{B}}{R_c^2 B^2}$

Drifts add. Bad news for toroidal confinement = bending field lines …

Direction of magnetic drifts in a torus

The poloidal field is much smaller than the toroidal field, $B_{pol} \sim \frac{1}{10} B_{tor}$

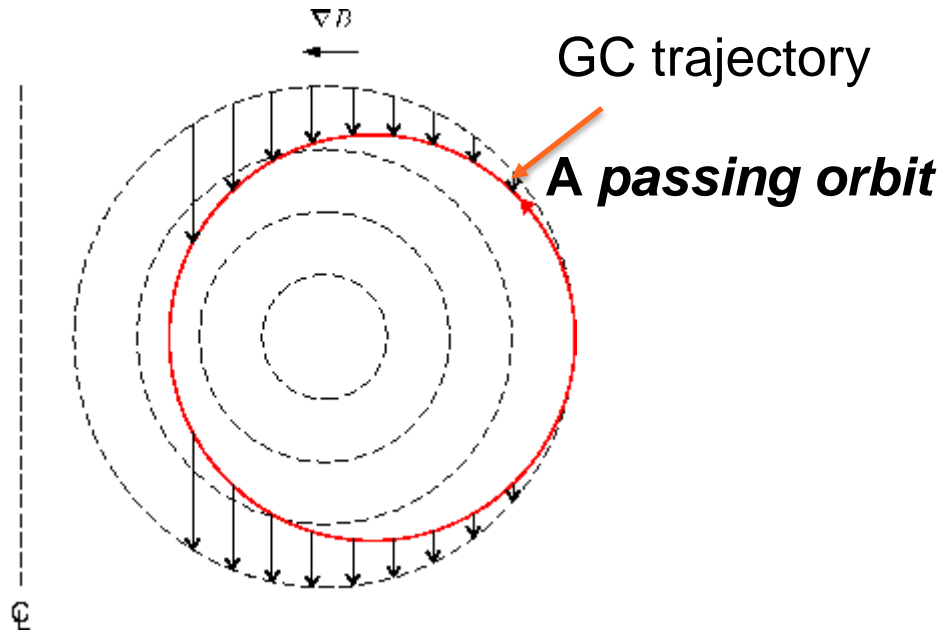➜ the direction of the drifts is dominated by B_{tor}

Gradient drift: $v_{\nabla B} \propto \boldsymbol{B} \times \nabla B \propto (\pm\hat{\phi}) \times (-\hat{R})$ ➜ gradient drift $\approx$ vertical !

Curvature drift: $\boldsymbol{v}_c \propto \boldsymbol{R}_c \times \boldsymbol{B} \propto \hat{R} \times (\pm\hat{\phi}) = (\pm\hat{\phi}) \times (-\hat{R})$

Thus it is further verified: gradient and curvature drifts add.

What do they do to our charged particles?

Drift orbits in toroidal geometry

GC trajectory

A *passing orbit*

∇B

Simon Pinches, PhD thesis

In this course the magnetic field strength can be approximated as

$$B = \frac{B_0 R_0}{R}$$

Taking into account how the particles move along the field lines, this can be expressed a little more informatively:

$$B = \frac{B_0 R_0}{R_0 + r \cos\theta}$$

With large *aspect ratio* $A = R_0/a \gg 1$

$$B = \frac{B_0}{1 + \epsilon \cos\theta} \approx B_0(1 - \epsilon \cos\theta), \epsilon = 1/A$$
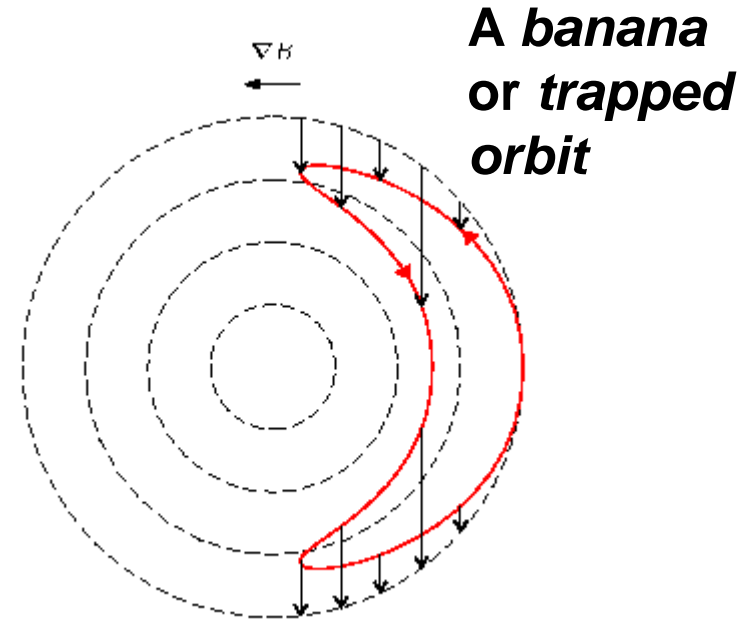
Trapping in toroidal geometry

1/R nature of the magnetic geometry ➔ field lines are denser at the inner part of the donuts

➔ there is a *gradient* in magnetic field towards the symmetry axis of the torus

➔ reflection can occur! ➔ HW

- Find the mirror point
- Fraction of banana particles

Width of this banana orbit:

poloidal Larmor radius $\Delta_b = mv_\parallel/qB_{pol} \sim 10r_L$

A *banana* or *trapped* orbit

Simon Pinches, PhD thesis

Drift orbits and transport

Remember the intuitive basis for diffusion via random walk:

$$D \propto (step\ length)^2 / \tau_{coll}$$

For classical diffusion: step length = Larmor radius

Now collisions can throw a particle from one banana orbit to another

➔ step length = *poloidal* Larmor radius

➔ so-called *neoclassical diffusion*, which is about $\frac{B_{tor}^2}{B_{pol}^2} \sim 100$ times larger !

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But we are still about a factor of 100 away from the observed transport …

More on 'anomalous' transport

More geometrical effects

"Previously on IPPFSA: … *misalignments of field coils can lead to stray field lines prematurely ending up at the wall*"

In toroidal geometry an additional loss channel, related to the field line topology, can appear:

Think: *how can we be sure that the drift orbit closes upon itself in the poloidal plane???*

Intuitive conclusion: *only if the drift in the upper and lower hemisphere are identical…*

But it is dangerous to blindly trust intuition. Luckily we have ….

Noether's theorem

Much of the physics you learned already in school is based on the work of a *mathematician*, Emmy Noether:

Any symmetry is accompanied by a conservation law

- Translational symmetry ➔ conservation of *linear momentum*
- Rotational symmetry ➔ conservation of *angular momentum*
- Symmetry in time ➔ conservation of *energy*

If our toroidal plasma is *axisymmetric*, the conserved quantity, called *toroidal canonical momentum*, is conserved and guarantees closing of the drift orbits!

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But…there is no such thing as an axisymmetric tokamak…

- The toroidal field is produced by $N \ll \infty$ external coils
- The extent of these coils is finite

➜ $R_{mirror} = R_{mirror}(\phi)$

➜ the banana orbit does not close upon itself

➜ banana orbit starts wandering off …

Notes:

- this effect is limited to the very edge of the toroidal plasma
  ➜ not responsible for global degradation of a plasma
- very small ξ_0 ➜ can even get trapped between two coils
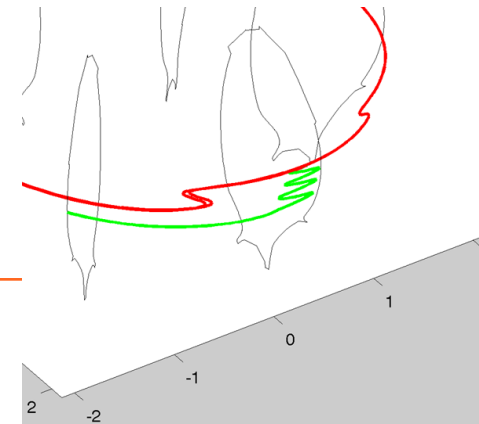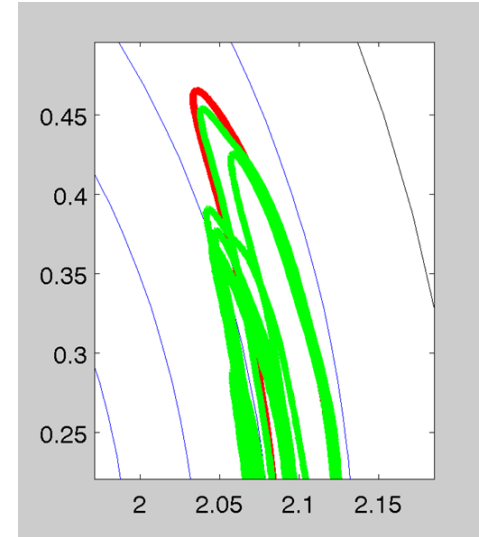
But what about non-uniform *E* field??

Requires straight-forward but complicated (= time consuming) math

➔ skip here. Will give the so-called *finite-Larmor-radius effects*:

$$v_E = \left(1 + \frac{1}{4}r_L^2\nabla^2\right)\frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2}$$

Note: the *ExB* drift is no longer independent of species due to r_L^2 !

➔ if a density clump appears… ➔ charge separation ➔ $\boldsymbol{E}$

A feedback mechanism ➔ a *drift instability* ➔transport

Note: important for large $k^2 r_L^2$, i.e., small wavelengths ➔ *microinstability*

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What about time-varying fields???

Time-varying E field …

Plasma is not a passive media but responds to changes (recall plasma oscillations) ➜ take uniform but sinusoidally varying field: $\boldsymbol{E} = E_0 e^{-i\omega t} \boldsymbol{x}$

➜ $\dfrac{dE_x}{dt} = i\omega E_x$ ➜ $\dfrac{d^2 v_x}{dt^2} = -\Omega_c^2 \left(v_x \mp \dfrac{i\omega}{\Omega_c} \dfrac{E_x}{B} \right)$; $(\boldsymbol{B} = B_0 \boldsymbol{z})$

Previous results ➜ define $v_E \equiv -\dfrac{E_x}{B}$ and $v_p \equiv \pm \dfrac{i\omega}{\Omega_c} \dfrac{E_x}{B}$

$$\frac{d^2 v_x}{dt^2} = -\Omega_c^2 \left(v_x - v_p \right)$$
$$\frac{d^2 v_y}{dt^2} = -\Omega_c^2 \left(v_y - v_E \right)$$

… and the *polarization drift*

From previous experience, let's make *trial solutions*

$$v_x = v_\perp e^{-i\Omega_c t} + v_p \qquad \frac{d^2 v_x}{dt^2} = -\Omega_c^2 \left(v_x - v_p \right) - \omega^2 v_p$$

$$v_y = \pm i v_\perp e^{-i\Omega_c t} + v_E \qquad \frac{d^2 v_y}{dt^2} = -\Omega_c^2 \left(v_y - v_E \right) - \omega^2 v_E$$

This is NOT the same as the original set of equations… ☹

However, IF $\omega^2 \ll \Omega_c^2$, then we can neglect the last term.

➔ For *sufficiently* slowly varying E-field the trial solutions are OK

➔ *Polarization drift* : $\boldsymbol{v}_p \equiv \pm \dfrac{1}{\Omega_c B} \dfrac{d\boldsymbol{E}}{dt}$

Physics of polarization drift etc

1. The polarization drift is *parallel* to the electric field!
2. Polarization drift depends on charge ➜ *polarization current !*

$$\boldsymbol{j}_p = \frac{\rho}{B^2} \frac{d\boldsymbol{E}}{dt} \text{ , where } \rho \text{ is the mass density of the plasma}$$

3. Polarization effect is similar to that in any dielectric where $\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P}$
   - In a plasma, the dipoles are formed by ions and electrons separated by the distance r_L
   - However, since ions and electrons are mobile and try to keep quasineutrality, a stationary electric field cannot sustain polarization current

Time-varying magnetic field …

We already know what happens in time-varying B field – at least if the variation is sufficiently slow: $\mu \equiv \frac{1}{2}\frac{mv_\perp^2}{B} = constant$

➔ Increasing B will increase $v_\perp$!   … But how much?

Time-varying magnetic field is associated with an electric field:

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

EoM ➔ $\frac{d}{dt}\left(\frac{1}{2}mv_\perp^2\right) = q\boldsymbol{E} \cdot \boldsymbol{v}_\perp = q\boldsymbol{E} \cdot \frac{d\boldsymbol{l}}{dt}$, where $\boldsymbol{v}_\perp = \frac{d\boldsymbol{l}}{dt}$.

Slowly varying field ➔ change over one gyro orbit: $\delta\left(\frac{1}{2}mv_\perp^2\right) = \oint q\boldsymbol{E} \cdot d\boldsymbol{l}$

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… and adiabatic compression = heating !

Use Stokes' theorem

$$\delta\left(\frac{1}{2}mv_\perp^2\right) = q\int(\boldsymbol{\nabla}\times\boldsymbol{E})\cdot d\boldsymbol{S} = -q\int\frac{\partial\boldsymbol{B}}{\partial t}\cdot d\boldsymbol{S}$$

Plasma is diamagnetic ➜ $\boldsymbol{B}\cdot d\boldsymbol{S} < 0$ for ions and > 0 for electrons

➜ $\delta\left(\frac{1}{2}mv_\perp^2\right) = \pm q\frac{\partial\boldsymbol{B}}{\partial t}\pi r_L^2 = \mu\frac{2\pi\frac{\partial\boldsymbol{B}}{\partial t}}{\Omega_c} = \mu\frac{1}{f_c}\frac{\partial\boldsymbol{B}}{\partial t}$; $\delta B_{orbit} = \frac{1}{f_c}\frac{\partial\boldsymbol{B}}{\partial t}$

➜ $\delta\left(\frac{1}{2}mv_\perp^2\right) = \mu\delta B_{orbit}$ &   additional result: $\mu = constant$ (… again)

So plasma can be heated by increasing B slowly: $v_\perp^2$ increases but r_L shrinks
➜ by *compressing* it.