Aalto University
School of Science

Lecture 6: Electromagnetic waves and more

Today's menu

- Cut-offs & Resonances
- Basic EM waves in plasmas:
  - Transverse EM wave in the absence of a background B-field
  - Ordinary wave (O-wave)
  - Extraordinary wave (X-wave)
  - L-wave
  - R-wave
- MHD waves:
  - Shear-Alfvén waves
  - Slow and fast magnetosonic waves (& compressional Alfvén wave)

Cut-offs and resonances

Cut-offs and resonances

For any non-trivial dispersive wave, there are two special cases

- $k \to 0$, i.e., $\lambda \to \infty$. This is the cut-off. A cut-off corresponds to a location where *reflection* (or strong *attenuation)* of the (EM) wave takes place
- $k \to \infty$, i.e., $\lambda \to 0$. This is called a *resonance*, and here the wave can be *absorbed* or *transferred* to another wave mode

Electromagnetic waves in vacuum

Allow time-dependent $\boldsymbol{E}$ and $\boldsymbol{B}$ fields → need Maxwell's equations

Maxwell's equations once again

$$\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0$$
$$\nabla \cdot \boldsymbol{B} = 0$$
$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$
$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{j} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}$$

In vacuum: $\rho = 0$, $\boldsymbol{j} = 0$

Wave equation in vacuum

Take curl of Faraday's law ➔

$$\nabla \times (\nabla \times \boldsymbol{E}) = -\frac{\partial}{\partial t}\nabla \times \boldsymbol{B} = -\frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

$$\nabla(\nabla \cdot \boldsymbol{E}) - \nabla^2 \boldsymbol{E} = -\frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

➔Basic wave equation: $\nabla^2 \boldsymbol{E} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0.$

Plane wave solution ➔ $k^2 \boldsymbol{E} - \frac{\omega^2}{c^2}\boldsymbol{E} = 0$ ➔ $\omega^2 = c^2 k^2$

Dispersion relation !

Wave equation in plasmas

In plasmas,

- $\rho \approx 0$ can be assumed by quasineutrality (or simply make the choice to look at $\boldsymbol{k} \cdot \boldsymbol{E} = 0$ since EM waves are ***usually*** transverse)
- $\boldsymbol{j} = 0$ is a very bad assumption.

➔ wave equation in plasmas: $\nabla^2 \boldsymbol{E} = \mu_0 \frac{\partial \boldsymbol{j}}{\partial t} + \frac{1}{c^2} \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$.

FT & linearize ➔ $(\omega^2 - c^2 k^2) \boldsymbol{E}_1 = -i \omega c^2 \mu_0 \boldsymbol{j}_1$

EM waves are fast ➔ ions immobile ➔ current solely from electrons

EM waves w/ no background B field

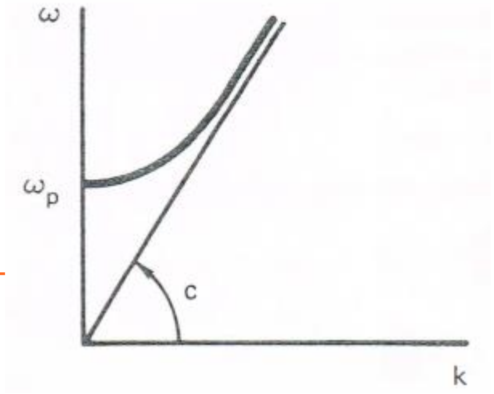
No guiding B field ➔ electrons are free to move: $m\frac{\partial \boldsymbol{v}}{\partial t} = -e\boldsymbol{E}$

FT & linearize ➔ $j_1 = -en_0\boldsymbol{v}_1 = -en_0\left(\frac{-e\boldsymbol{E}_1}{-i\omega m}\right) = i\,\frac{e^2 n_0}{m\omega}\boldsymbol{E}_1$

➔ $(\omega^2 - c^2k^2)\boldsymbol{E}_1 = -i\omega c^2\mu_0\boldsymbol{j}_1 = \cancel{\omega}c^2\mu_0\frac{e^2n_0}{m\cancel{\omega}}\boldsymbol{E}_1 = \frac{e^2n_0}{m\epsilon_0}\boldsymbol{E}_1 = \omega_p^2\boldsymbol{E}_1$

➔ Dispersion relation for transverse EM waves propagating in plasmas in the absence of DC magnetic field:

$$\omega^2 = \omega_p^2 + c^2k^2$$



Observations on the dispersion relation

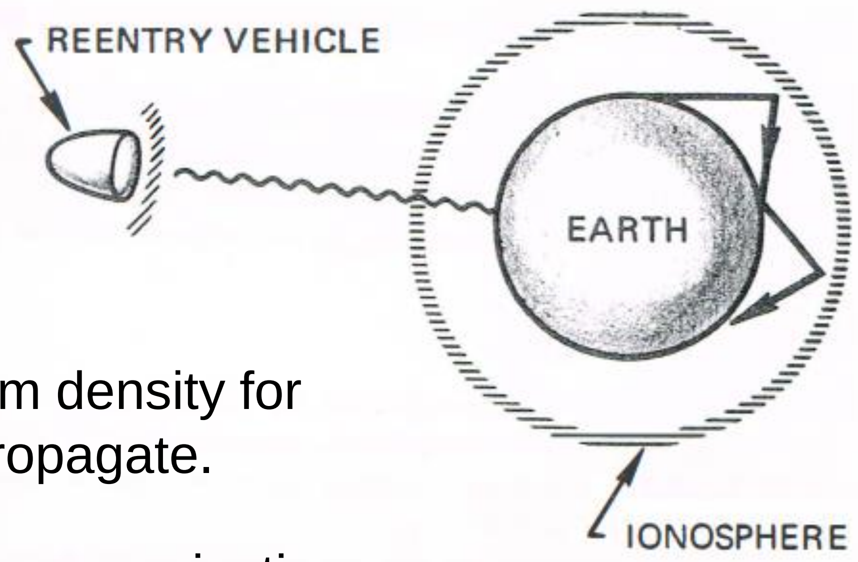
1. $v_{ph}^2 = \frac{\omega^2}{k^2} = c^2 + \frac{\omega_p^2}{k^2} > c^2$
2. $v_{gr} = \frac{d\omega}{dk} = \frac{c^2}{v_{ph}} < c$
3. At large k (small λ) ➔ ordinary light waves, $\omega = ck$
4. There is a *cut-off frequency* for waves to propagate …

$$\omega < \omega_{cut-off} = \omega_p \Rightarrow ck = \sqrt{\omega^2 - \omega_p^2} = i\sqrt{\omega_p^2 - \omega^2}$$

➔ $e^{ikx} = e^{-Im(k)x} = e^{-\frac{x}{\delta}}$, where $1/\delta \equiv \frac{1}{c}\sqrt{\omega_p^2 - \omega^2}$,

➔ an exponentially attenuated wave with skin depth δ

Radio communication



Recall $\omega_p^2 = \frac{e^2 n_0}{m\epsilon_0}$

➔ For a given frequency ω, there is a maximum density for plasmas through which the wave can still propagate.

- This is the basis of short-wavelength radio communication
- To communicate with a satellite, the wave frequency has to be chosen sufficiently high to penetrate all atmospheric layers
- Space vehicle entering the atmosphere will suffer a communication black-out due to the shock wave in front of it

Electromagnetic waves with $B_0 \neq 0$
$k \perp B_0$

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Transverse waves propagating perpendicular to B_0 : *ordinary wave*

Transverse wave: $\boldsymbol{k} \perp \boldsymbol{E}$

Propagation perpendicular to magnetic field: $\boldsymbol{k} \perp \boldsymbol{B_0}$

#1. Take $\boldsymbol{E}_1 \parallel \boldsymbol{B_0}$

Then the magnetic field does not constrain the electron motion and the math of $\boldsymbol{B_0} = 0$ case applies

$$\omega^2 = \omega_p^2 + c^2 k^2$$

This is called an *ordinary wave* – or just *O-wave* between friends.

Finding the *extraordinary* wave

#2. Take $\boldsymbol{E}_1 \perp \boldsymbol{B}_0$

Now electron motion *is* constrained by $\boldsymbol{B}$.

Take x-axis so that $\boldsymbol{k} = k\hat{\boldsymbol{x}}$ and $\boldsymbol{E}_1 = E_x\hat{\boldsymbol{x}} + E_y\hat{\boldsymbol{y}}$

- It is known that in this case a longitudinal component will arise ➔ $E_x \neq 0$.

Electron EoM, linearized and FT'd:

$$-im\omega\boldsymbol{v}_1 = -e(\boldsymbol{E}_1 + \boldsymbol{v}_1 \times \boldsymbol{B}_0)$$

$$v_x = \frac{-ie}{m\omega}\left(E_x + v_y B_0\right) \qquad \qquad v_x = \frac{e}{m\omega}\left[-iE_x - \frac{\Omega_e}{\omega}E_y\right] \Big/ \left(1 - \frac{\Omega_e^2}{\omega^2}\right)$$

$$v_y = \frac{-ie}{m\omega}\left(E_y - v_x B_0\right) \qquad ➔ \qquad v_y = \frac{e}{m\omega}\left[-iE_y + \frac{\Omega_e}{\omega}E_x\right] \Big/ \left(1 - \frac{\Omega_e^2}{\omega^2}\right)$$

Dispersion relation for the *X* wave

Now careful with the wave equation: $E_x \neq 0$ ➔ $\boldsymbol{k} \cdot \boldsymbol{E} \neq 0$

➔ $(\omega^2 - c^2k^2)\boldsymbol{E}_1 + c^2 k E_x \boldsymbol{k} = -i\omega c^2 \mu_0 \boldsymbol{j}_1 = i\omega c^2 \mu_0 n_0 e \boldsymbol{v}_1$

We already have $\boldsymbol{v}_1 = \boldsymbol{v}_1(\boldsymbol{E}_1)$ ➔

A matrix equation: $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix} = 0$

➔Use the determinant condition to find the non-trivial solution …

HW ➔ $\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2} \frac{\omega^2 - \omega_p^2}{\omega^2 - \omega_h^2}$, dispersion relation for the X-wave

Cut-offs and resonances

We have just obtained our first *non-trivial* dispersion relation.

In particular, it can happen that

- $k \to 0$, i.e., $\lambda \to \infty$. This is the cut-off that we already got for $\boldsymbol{B_0} = 0$. A cut-off corresponds to *reflection* of the EM wave
- $k \to \infty$, i.e., $\lambda \to 0$. This is called a *resonance*, and here the wave can be *absorbed*.

How do cut-offs and resonances look for the *X*-wave?

Cut-offs and resonances of the *X*-wave

- Resonance:
  - $k \to \infty$ when $\omega \to \omega_h$

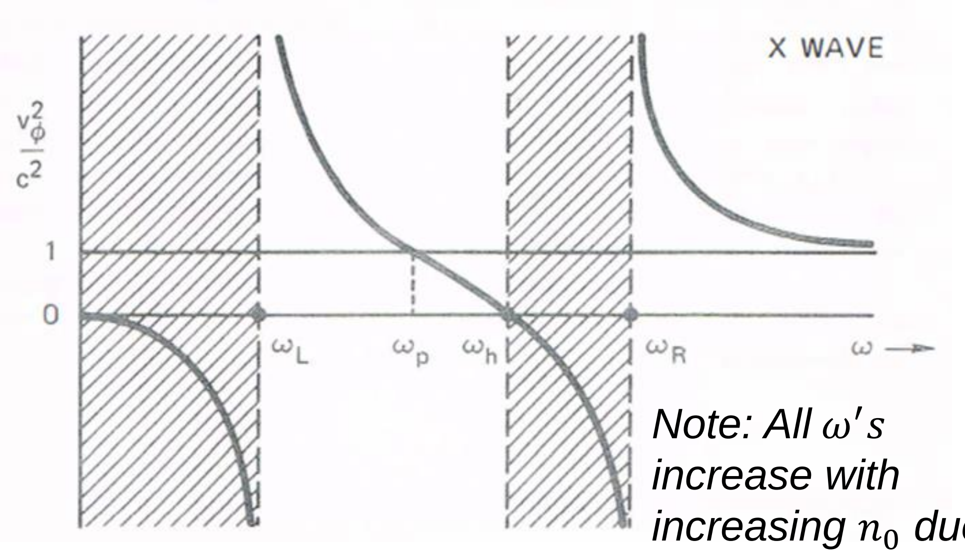
  ➔ Resonance occurs at a point in the plasma where $\omega^2 = \omega_h^2 = \omega_p^2 + \Omega_e^2$

  But this dispersion relation we know: *electrostatic 'waves' across* B_0 !

  ➔ When an EM wave approaches a point in a plasma where $\omega \to \omega_h$, both v_{ph} and v_{gr} go to zero and the wave is converted into *upper hybrid oscillations* !

- Cut-off:
  - $k \to 0$ when $1 = \frac{\omega_p^2}{\omega^2}\frac{\omega^2-\omega_p^2}{\omega^2-\omega_h^2} = \frac{\omega_p^2}{\omega^2}\left[1 - \frac{\Omega_e^2}{\omega^2-\omega_p^2}\right]^{-1}$ $\xrightarrow{\text{HW}}$ $\omega^2 \mp \Omega_e\omega - \omega_p^2 = 0$

Stop bands for *X*-wave



HW ➔ 2 cut-off frequencies:

$$\omega_R = \frac{1}{2}\left[\sqrt{\Omega_e^2 + 4\omega_p^2} + \Omega_e\right]$$

$$\omega_L = \frac{1}{2}\left[\sqrt{\Omega_e^2 + 4\omega_p^2} - \Omega_e\right]$$

Note: All ω's increase with increasing n_0 due to ω_p dependence

The resonance and cut-off frequencies divide the dispersion diagram into propagation and non-propagation zones.

➔ *X*-wave has two regions of propagation, separated by a ’stop band’ where it cannot propagate.

The new *dispersion diagram*

Note: the dispersion diagram for the *X*-wave was no longer of the type $\omega = \omega(k)$.

The reason is that we do not have simple enough functional dependence between ω and k.

➔ Plotting $\frac{\omega}{ck} = \frac{v_{ph}}{c}$ as a function of ω has proven to be enlightening.

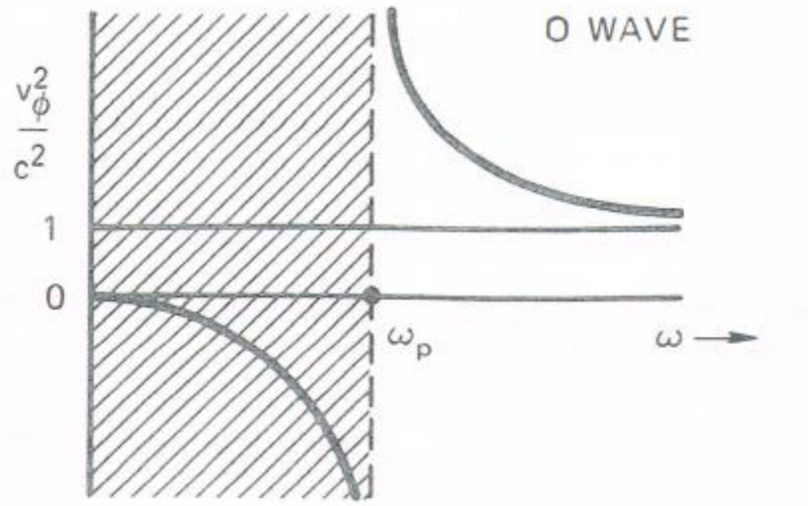
Stop band for the *O* wave?

Simpler: $\omega^2 = \omega_p^2 + c^2k^2$

➔ $$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}$$

➔ No resonances

➔ One cut-off: $k \to 0$ when $\omega = \omega_p$ (as was discovered already ☺)



Electromagnetic waves with $B_0 \neq 0$
$k \parallel B_0$

How about waves parallel to $\boldsymbol{B_0}$?

Now $\boldsymbol{k} \parallel \boldsymbol{B_0}$ ➔ $\boldsymbol{k} = k\hat{\boldsymbol{z}}$, and from electron motion we can expect $\boldsymbol{E}_1 = E_x\widehat{\boldsymbol{x}} + E_y\widehat{\boldsymbol{y}}$ ➔ we can use the wave equation from X wave with the substitutions $\boldsymbol{k} = k\widehat{\boldsymbol{x}} \rightarrow k\hat{\boldsymbol{z}}$ ➔

$$(\omega^2 - c^2k^2)E_x = \alpha\left(E_x - \frac{i\Omega_e}{\omega}E_y\right), \quad \text{where } \alpha = \frac{\omega_p^2}{1-\Omega_e^2/\omega^2}$$

$$(\omega^2 - c^2k^2)E_x = \alpha\left(E_x + \frac{i\Omega_e}{\omega}E_y\right)$$

Again we have a coupled set of equations ➔ use det = 0 ➔

$$\omega^2 - c^2k^2 - \alpha = \pm\alpha\frac{\Omega_e}{\omega}$$

Wave names by polarization

We then obtain two waves propagation along the B field:

- R-wave: $\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}\left[1 - \frac{\Omega_e}{\omega}\right]^{-1}$
- L-wave: $\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}\left[1 + \frac{\Omega_e}{\omega}\right]^{-1}$

Reason for names:

The $\boldsymbol{E}_1$vector of the *R*-wave rotates clockwise in time as viewed in the direction of propagation ➔ *right-hand circularly polarized wave*

Vice versa for the *L*-wave.

Cut-offs and resonances for *L* and *R* ?

Resonances:

- R-wave: $k \rightarrow \infty$ @ $\omega = \Omega_e$, giving a resonance. Physics of the resonance: polarization allows the E field to be in sync with the electron gyration ➔ wave dumps its energy to electrons ➔ *electron cyclotron resonance heating (ECRH)*
- L-wave: no resonance found. (would exist if ion motion were included)

Cut-offs:

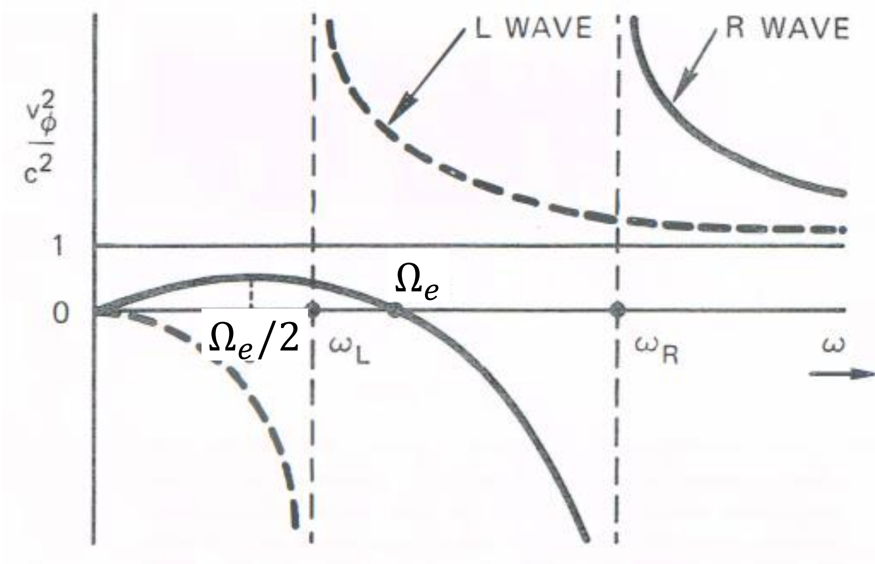
- R-wave: $k \rightarrow 0$ @ $\omega = \omega_R$.
- L-wave: $k \rightarrow 0$ @ $\omega = \omega_L$.

The names of the frequencies make sense! ☺

Stop bands for *R* and *L* waves

The *L*-wave

- Has a stop band at low- ω ➔ behaves like an O-wave except with replacement $\omega_p \rightarrow \omega_L$



The *R*-wave

- Has a stop band between $[\Omega_e, \omega_R]$
- The low-frequency band, $\omega < \Omega_e$, has an interesting history and relevance

➔ HW: *Food for Thought*

Summary of EM waves in plasmas

Along the B field:

- Right-hand (*R*) and left-hand (*L*) circularly polarized waves

Across the B-field:

- plane-polarized ordinary (*O*) wave and elliptically polarized extraordinary (*X*) wave

Magnetohydrodynamic waves

What is different now?

- Until now we have always been aware that plasma consists of ions and electrons ➔ we have made choices of which dynamics to include.
- In magnetohydrodynamics, the plasma is just a *fluid*

➔ in MHD, the waves are supported/carried by plasma fluid, where the ion and electron species have just as much to say as oxygen and hydrogen have in regular *hydro*dynamics.

We shall now apply our procedure to the MHD equations …

Linearized MHD equations

Do the linearization procedure for the MHD equations ➔

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \boldsymbol{V_1} = 0$$

$$\rho_0 \frac{\partial \boldsymbol{V}_1}{\partial t} + \nabla p_1 - \frac{(\nabla \times \boldsymbol{B}_1) \times \boldsymbol{B}_0}{\mu_0} = 0$$

$$-\frac{\partial \boldsymbol{B}_1}{\partial t} + \nabla \times (\boldsymbol{V}_1 \times \boldsymbol{B}_0) = 0$$

$$\frac{\partial}{\partial t}\left(\frac{p_1}{p_0} - \frac{\gamma \rho_1}{\rho_0}\right) = 0$$

here ρ is the fluid density

”Plane wave” solution …

Now take the harmonic approximation and assume that each perturbed quantity is a sum of plane waves w/ given $\boldsymbol{k}$ and ω :

$$-\omega\rho_1 + \rho_0 \boldsymbol{k}\cdot\boldsymbol{V}_1 = 0 \Rightarrow \rho_1 = \rho_0 \frac{\boldsymbol{k}\cdot\boldsymbol{V}_1}{\omega}$$

$$\omega \boldsymbol{B}_1 + \boldsymbol{k}\times(\boldsymbol{V}_1\times\boldsymbol{B}_0) = 0 \Rightarrow \boldsymbol{B}_1 = \frac{(\boldsymbol{k}\cdot\boldsymbol{V}_1)\,\boldsymbol{B}_0 - (\boldsymbol{k}\cdot\boldsymbol{B}_0)\,\boldsymbol{V}_1}{\omega}$$

$$-\omega\left(\frac{p_1}{p_0} - \frac{\gamma\rho_1}{\rho_0}\right) = 0 \Rightarrow p_1 = \gamma p_0 \frac{\boldsymbol{k}\cdot\boldsymbol{V}_1}{\omega}$$

$$-\omega\rho_0\boldsymbol{V}_1 + \boldsymbol{k}p_1 - \frac{(\boldsymbol{k}\times\boldsymbol{B}_1)\times\boldsymbol{B}_0}{\mu_0} = 0$$

4 equations, 4 unknowns!

Substitute the expressions for $\boldsymbol{B}_1$ and p_1 ➔ equation for $\boldsymbol{V}_1$:

$$\left[\omega^2 - \frac{(\boldsymbol{k}\cdot\boldsymbol{B}_0)^2}{\mu_0\rho_0}\right]\boldsymbol{V}_1 =$$

$$\left\{\left[\frac{\gamma p_0}{\rho_0} + \frac{B_0^2}{\mu_0\rho_0}\right]\boldsymbol{k} - \frac{(\boldsymbol{k}\cdot\boldsymbol{B}_0)}{\mu_0\rho_0}\boldsymbol{B_0}\right\}(\boldsymbol{k}\cdot\boldsymbol{V}_1)$$

$$-\frac{(\boldsymbol{k}\cdot\boldsymbol{B}_0)(\boldsymbol{V}_1\cdot\boldsymbol{B}_0)}{\mu_0\rho_0}\boldsymbol{k}$$

Get real(istic) …

Time to fix the space:

- Align the coordinate system so that $\boldsymbol{B}_0 = B_0\boldsymbol{z}, \boldsymbol{k} = k_x\boldsymbol{x} + k_z\boldsymbol{z}$,
- Angle θ defined to be the angle between $\boldsymbol{k}$ and $\boldsymbol{B}_0$

Write the linearized equation of motion in (x, y, z) components:

➔ Matrix equation

$$\begin{pmatrix} \omega^2 - k^2 V_A^2 - k^2 V_S^2 \sin^2\theta & 0 & -k^2 V_S^2 \sin\theta\cos\theta \\ 0 & \omega^2 - k^2 V_A^2 \cos^2\theta & 0 \\ -k^2 V_S^2 \sin\theta\cos\theta & 0 & \omega^2 - k^2 V_S^2 \cos^2\theta \end{pmatrix} \begin{pmatrix} V_{1x} \\ V_{1y} \\ V_{1z} \end{pmatrix} = 0,$$

Here $V_A^2 = \frac{B_0^2}{\mu_0 \rho_0}$ is the so-called *Alfvén speed*, and $V_S^2 = \frac{\gamma p_0}{\rho_0}$ is the *sound speed*

Non-trivial ($\neq 0$) solutions only for det = 0

➔ Product of three terms (easiest by using either middle row or middle column)

➔ $$\left(\omega^2 - k^2 V_A^2 \cos\theta\right)\left[\omega^4 - \omega^2 k^2 \left(V_A^2 + V_S^2\right) + k^4 V_A^2 V_S^2 \cos^2\theta\right] = 0$$

➔ 3rd order equation for ω^2

➔ Different kinds of *Alfvén waves* …

Shear Alfvén wave

$$\omega^2 - k^2 V_A^2 \cos^2\theta = 0$$

This corresponds to $\boldsymbol{k} \cdot \boldsymbol{V}_1 = 0$

➔ No density or pressure perturbation associated with the wave

Also $\boldsymbol{V}_1 \cdot \boldsymbol{B}_0 = 0$

➔ Motion only perpendicular to the magnetic field

Compressional Alfvén wave

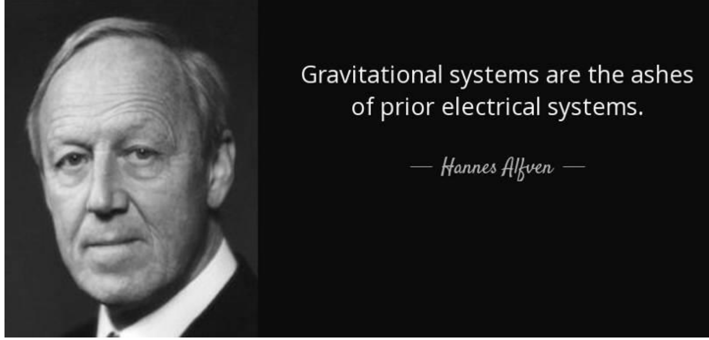
The other two roots: $\omega = kV_{\pm}$, where

$$V_{\pm} = \left\{ \frac{1}{2} \left[V_A^2 + V_S^2 \pm \sqrt{\left(V_A^2 + V_S^2\right)^2 - 4V_A^2 V_S^2 \cos^2\theta} \right] \right\}^{1/2}$$

These are the fast and slow *magnetosonic* waves – or *fast* and *slow* waves between friends.

- In the cold plasma limit ($p \rightarrow 0$), the *fast* wave becomes the so-called *compressional Alfven wave*: $\omega = kV_A$ and slow wave dies.
- For $V_A \ll V_S$ the *fast* wave becomes a sound wave, modified by the presence of the magnetic field: $\omega = kV_S$

Hannes Alfvén (1908 – 1995)



- developed the MHD theory
- was the first to discover these wave motions within MHD
- Nobel prize 1970

Alfvén waves can be used to diagnose the plasma (especially in space, but also in laboratory plasmas)