Robust optimization

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Simple optimization problem

***An original, novel problem*: Jimmy needs help fulfilling several nutritional goals while minimizing the total costs.**

$$P: min.\, c^T x$$
$$s.t.$$
$$Ax \geq b$$
$$x_i \geq 0\ \forall i$$

X* =

sweet potatoes	apples	soybeans	carrots	almonds
2000 g	0 g	200 g	0 g	0 g

Cost: 4.55€/day

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Simple optimization problem

After some googling, Jimmy finds out that the calcium, and vitamin A value of sweet potatoes can vary by as much as 50% depending on the type of potato and other factors.

Out of a sample of 10 sweet potatoes, 8 would cause Jimmy to have a severe deficiency of either of the nutrients.

A new formulation is needed.

Simple optimization problem

New formulation:

$$
\begin{aligned}
\boldsymbol{P} \quad &: \boldsymbol{min.\, c^T x} \\
&\boldsymbol{s.t.} \\
&\left(\boldsymbol{A_0 + d_{potato} A_{potato}}\right)\boldsymbol{x \geq b}\ \forall \boldsymbol{d_{potato}} \quad \in \boldsymbol{D} \\
&\boldsymbol{x} \quad \geq \boldsymbol{0}
\end{aligned}
$$

X* =

sweet potatoes	apples	soybeans	carrots	Almonds
1800 g	0 g	180 g	250 g	50 g

Cost: 4.55€/day => 5.08€/day

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Content

1. Motivation
2. Definition
3. Approaches
4. Conclusion

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But wait!

This seems familiar…

Presentations 1-10: Optimization under uncertainty

This is just stochastic optimization!

NO

Robust optimization vs. stochastic optimization

Probability distribution often unknown

Stochastic optimization problems can be NP-hard for seemingly simple problems

Scenario sets can be difficult to construct

Assumptions

1. **Decisions are here-and-now**

   Johnny can only afford one nutritional expert consultation

2. **Constraints can't be violated**

3. **Uncertainty set contains all possible realizations**

   Set of possible nutrient values for potatoes

Traditional robust optimization

$$\begin{aligned}&P: min.\, c^T x\\&s.t.\\&Ax \leq b\ \forall A \in \mathbf{U}\\&x \geq 0\end{aligned}$$

Equivalent to

$$\begin{aligned}&min.\, c^T x\\&s.t.\\&\bar{A}x \leq b\\&x \geq 0\end{aligned}$$

Where $\overline{A_{ij}} = sup_{A \in U}\left(A_{ij}\right)$ (Soyster 1973)

Note: Only the case in non-adjustable cases!

Traditional robust optimization

Benefits

- **Guarantees feasibility for all realizations**
- **Minimizes cost in worst-case scenario**
- **Simple to interpret**

Weaknesses

- **Gives overly conservative solutions**
- **Tractability depends on uncertainty set U**

Robust counterpart

$\mathbf{LO_u}: \{min_x \{c^T x + d : Ax \leq b\} \}\forall(c,d,A,b) \in \mathbf{U}$

A solution is robust feasible if it satisfies all possible constraints $Ax \leq b \; \forall \; (A,b) \in \mathbf{U}$

For a solution x, the robust value is the largest value in the uncertainty set: $\hat{c}(x) = sup_{(c,d,A,b)\in\mathbf{U}} \; c^T x + d$

RC: $min_x \{sup_{c,d,A,b\in\mathbf{U}}[c^T x + d]: Ax \leq b \; \forall(c,d,A,b) \in \mathbf{U}\}$

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Robust counterpart

$\mathbf{LO_u}: \{min_x \{c^T x + d : Ax \leq b\}\} \forall (c, d, A, b) \in \mathrm{U}$

RC: $min_x \{sup_{c,d,A,b \in \mathrm{U}} [c^T x + d]: Ax \leq b \; \forall (c, d, A, b) \in \mathrm{U}\}$

Note: uncertainty in objective function can always be represented as uncertainty in constraints!

$$min_{x,t} \left\{ t: \begin{matrix} c^T x - t \leq -d \\ Ax \leq b \end{matrix} \; \forall (c, d, A, b) \in U \right\}$$

Uncertainty sets

Robust counterpart of a problem is always tractable, when the uncertainty set is tractable

If x is a robust feasible solution to a constraint $(\boldsymbol{C_i})$: $\boldsymbol{a_i^T x \leq b_i}$ with uncertainty set $(\boldsymbol{a_i}, \boldsymbol{b_i}) \in \boldsymbol{U_i}$, it's also robust feasible for the closed convex hull of the uncertainty set (Ben-Tal et al. 2009)

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Perturbation set

For linear problems, the uncertainty can be described as a perturbation set $\mathbf{Z}$ and basic shifts $\boldsymbol{A_i}, \boldsymbol{i} \in [\mathbf{1}, \boldsymbol{I}], \boldsymbol{b_j}, \boldsymbol{j} \in [\mathbf{1}, \boldsymbol{J}]$, and the problems as

$$\boldsymbol{P}: \boldsymbol{min}\ \boldsymbol{c^T x} + \boldsymbol{d}$$

$$\boldsymbol{s.t.}$$

$$\left(\boldsymbol{A_o} + \sum_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{I}} \boldsymbol{\zeta_i A_i}\right)\boldsymbol{x} \le \boldsymbol{b_o} + \sum_{\boldsymbol{j}=\mathbf{1}}^{\boldsymbol{J}} \boldsymbol{\zeta_j b_j}$$

$$\boldsymbol{\zeta_i} \in \mathbf{Z_i}, \boldsymbol{\zeta_j} \in \mathbf{Z_j}$$

Simple optimization problem

$$A_1 =$$

	sweet potatoes	apples	soybeans	carrots	Almonds
Protein	0	0	0	0	0
Fiber	0	0	0	0	0
Calcium	10 mg	0	0	0	0
Vitamin A	0	0	0	0	0

$$\zeta_1 \in [-1, 1]$$

$$A_2 =$$

	sweet potatoes	apples	soybeans	carrots	Almonds
Protein	0	0	0	0	0
Fiber	0	0	0	0	0
Calcium	0	0	0	0	0
Vitamin A	150 µg	0	0	0	0

$$\zeta_2 \in [-1, 1]$$

Perturbation set

Often the perturbations are given constraint-wise

$$\left(\boldsymbol{a}_{\boldsymbol{0}}^{\boldsymbol{T}} + \sum_{\boldsymbol{i=1}}^{\boldsymbol{I}} \boldsymbol{\zeta_i a_i^T}\right)\boldsymbol{x} \leq b_o + \sum_{\boldsymbol{j=1}}^{\boldsymbol{J}} \zeta_j b_j$$

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Uncertainty sets

The benefits of robust optimization are largely determined by the quality of the uncertainty sets.

Table 1 Tractable reformulations for the uncertain constraint $[(\mathbf{a^0}+\mathbf{P}\boldsymbol{\zeta})^\top \mathbf{x} \leq \beta \quad \forall \boldsymbol{\zeta} \in \mathcal{Z}]$, and h_k^* is the convex conjugate of h_k

Uncertainty	$\mathcal{Z}$	Robust Counterpart	Tractability
Box	$\lVert\boldsymbol{\zeta}\rVert_\infty \leq \rho$	$(\mathbf{a^0})^\top \mathbf{x} + \rho \lVert\mathbf{P}^\top \mathbf{x}\rVert_1 \leq \beta$	LP
Ellipsoidal	$\lVert\boldsymbol{\zeta}\rVert_2 \leq \rho$	$(\mathbf{a^0})^\top \mathbf{x} + \rho \lVert\mathbf{P}^\top \mathbf{x}\rVert_2 \leq \beta$	CQP
Polyhedral	$\mathbf{D}\boldsymbol{\zeta} + \mathbf{d} \geq 0$	$\begin{cases} (\mathbf{a^0})^\top \mathbf{x} + \mathbf{d}^\top \mathbf{y} \leq \beta \\ \mathbf{D}^\top \mathbf{y} = -\mathbf{P}^\top \mathbf{x} \\ \mathbf{y} \geq \mathbf{0} \end{cases}$	LP
Convex cons.	$h_k(\boldsymbol{\zeta}) \leq 0 \quad \forall k$	$\begin{cases} (\mathbf{a^0})^\top \mathbf{x} + \sum_k u_k h_k^*\left(\frac{\mathbf{w^k}}{u_k}\right) \leq \beta \\ \sum_k \mathbf{w^k} = \mathbf{P}^\top \mathbf{x} \\ \mathbf{u} \geq \mathbf{0} \end{cases}$	Convex Opt.

(Gorissen et al. 2015)

Introducing probability

The previous formulations lead to very conservative formulations

Can we use knowledge of a probability distribution to get better results?

Can we be certain that the uncertainty sets contain all possible values?

S7: Chance constraints

Chance constraints

$$\boldsymbol{P}\left\{\left(\boldsymbol{a}_{\mathbf{0}}^{T}+\sum_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{I}}\boldsymbol{\zeta}_{\boldsymbol{i}}\boldsymbol{a}_{\boldsymbol{i}}^{T}\right)\boldsymbol{x}\leq b_{o}+\sum_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{I}}\zeta_{i}b_{i}\right\}\geq 1-\epsilon$$

$$\boldsymbol{\zeta_i}\in\mathbf{Z}\sim\boldsymbol{P}$$

For independent *P*$=\mathrm{U}(-1,1)$ the above problem in NP-hard

(Khachiyan 1989)

Feasible sets often non-convex

Exact form of *P* often not known => *distributionally robust models*

Safe tractable approximations

***S*, a system of convex constraints on x and additional variables *v* s a safe tractable approximation of a chance costraint, if the x-component of every feasbile solution (x,v) satisfies the chance constraint.**

(Ben-Tal et al. 2009)

Chance constraints

$$P\left\{\left(a_0^T + \sum_{i=1}^{I} \zeta_i a_i^T\right) x \le b_o + \sum_{i=1}^{I} \zeta_i b_i\right\} \ge 1 - \epsilon$$

$$\zeta_i \in \mathrm{Z} = \mathrm{U}(-1, 1)$$

Safe tractable approximation:

$$\mathrm{P}\left\{\Omega \sqrt{\sum_{i=1}^{I} (a_i^T x - b_i)^2} \le b_o - a_0^T x\right\} \ge \exp(-\frac{\Omega^2}{2})$$

(Ben-Tal et al. 2009)

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Globalized robust counterpart

$$P: \min c^T x + d$$
$$s.t.$$
$$\left(A_o + \sum_{i=1}^{I} \zeta_i A_i\right) x - b \leq \alpha \cdot dist(\zeta, \mathrm{Z}) \forall \zeta \in \mathrm{Z}^+$$

Z : ”Normal uncertainty set”

Z^+: Possible uncertainty set

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Conclusions

- Uncertainty is prevalent in optimization problems, but often no probability distribution can be assigned
- Worst-case scenario optimization is often sensible
- Chance-constrained stochastic models can be difficult to solve, robust counterpart formulations can give good, tractable approximations

Sources

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