Adjustable robust optimization

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Remember the farmer?

- Crop planting decisions
- Probabilities from experience
- Willing to take some risk

→ Stochastic optimization!

- Uncertain data
- No probability information
- Risk-averse decisions

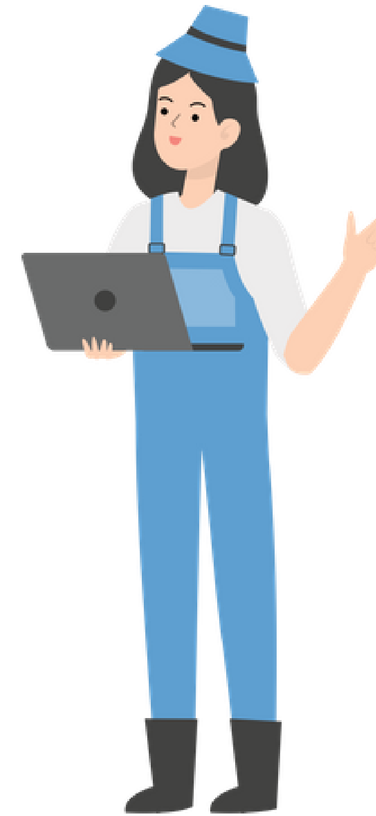
→ Robust Optimization!

Robust Optimization

+ does not need probability distributions
+ risk-averse
+ tractable

− no 2nd stage decisions
− overprotective (conservative)

→ **Solution: ARO**



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Adjustable Robust Optimization

Stochastic Programming vs ARO

SP: $\min \; c^T x \; + \; E_\zeta \; [\, d^T y(\zeta) \,]$

ARO: $\min \; c^T x \; + \; \max_\zeta \; [\, d^T y(\zeta) \,]$

Adjustable Robust Optimization

Ben-Tal et al. (2004)

1st stage decisions

"worst case"

$$\min \quad c^T x + \max_{\zeta} \; d^T y(\zeta)$$

$$\text{s.t.} \quad A_{\zeta} x + B_{\zeta} y(\zeta) \le b_{\zeta} \qquad \forall \zeta \in Z$$

2nd stage decisions

- 2nd stage decisions $y(\zeta)$ are functions of the uncertainty
- Objective function looks at the worst case scenario
- Adjustability counteracts overprotectiveness
- Easy(ish) to compute for infinitely many scenarios!

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Example:

- Want to grow apple and cherry trees
- Must reserve land to plant them
- One tree yields 50kg of apples/cherries
- Birds eat 50kg of combined fruit, preference tbd
- Want at least 100kg of apples and 100kg of cherries

Example:

Number of trees

$$\min \quad x$$

Apples (kg)

Cherries (kg)

s.t. $y_1(\zeta) + y_2(\zeta) \leq 50x \quad \forall \zeta \in [0, 50]$ (apple + cherry ≤ total fruit)

$y_1(\zeta) \geq 100 + \zeta \quad \forall \zeta \in [0, 50]$ (enough apples)

$y_2(\zeta) \geq 100 + (50 - \zeta) \quad \forall \zeta \in [0, 50]$ (enough cherries)

$x \geq 0$

$y_1 = 150$
$y_2 = 150$

$x = 6$

$y_1(\zeta) = 100 + \zeta$
$y_2(\zeta) = 150 - \zeta$

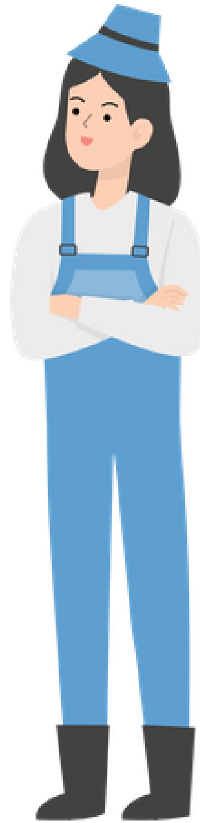
$x = 5$

Tractability

- **Complicated 2nd stage variables**
- **Tractable in some cases**
- **In general: NP-hard**

  → RO tractability advantage is gone!

- **Solution: Approximate!**



Affine Adjustability

Affine Adjustability

Ben-Tal et al. (2004)

- **Recourse decisions are functions**
- **Instead of arbitrary functions, use a certain class**
- **Affine functions:** $x \rightarrow Wx + w$
- **Replace** $y(\zeta)$ **with** $W\zeta + w$
- **2 normal variables instead of functions**

$$\min \quad c^T x + \max_\zeta \; d^T (W^T \zeta + w)$$

$$\text{s.t.} \quad A_\zeta x + B_\zeta (W\zeta + w) \leq b_\zeta \quad \forall \zeta \in Z$$

Example

$$\min \quad x$$

$$\text{s.t.} \quad y_1(\zeta) + y_2(\zeta) \le 50x \qquad \forall \zeta \in [0, 50]$$

$$y_1(\zeta) \ge 100 + \zeta \qquad \forall \zeta \in [0, 50]$$

$$y_2(\zeta) \ge 100 + (50 - \zeta) \qquad \forall \zeta \in [0, 50]$$

$$x \ge 0$$

Example

$$\min \quad x$$

$$\text{s.t.} \quad (W_1 \zeta + w_1) + (W_2 \zeta + w_2) \leq 50x \quad \forall \zeta \in [0, 50]$$

$$(W_1 \zeta + w_1) \geq 100 + \zeta \quad \forall \zeta \in [0, 50]$$

$$(W_2 \zeta + w_2) \geq 100 + (50 - \zeta) \quad \forall \zeta \in [0, 50]$$

$$x \geq 0$$

$$y_1(\zeta) = \zeta + 100 \qquad (W_1, w_1) = (1, 100)$$

$$y_2(\zeta) = 150 - \zeta \qquad (W_2, w_2) = (-1, 150)$$

Limitations

$$\min \quad c^T x + \max_{\zeta} \; d^T (W^T \zeta + w)$$

$$\text{s.t.} \quad A_{\zeta} x + B_{\zeta} (W\zeta + w) \leq b_{\zeta} \quad \forall \zeta \in Z$$

- Computationally tractable in many cases
- Optimal in some cases
- Possibly intractable for non-fixed recourse
- No integer 2nd stage decisions

But we can't plant half of an apple tree!

Column-and-Constraint Generation Method

CCG Method for ARO

Zeng & Zhao (2013)

$$\min \quad c^T x + \max_\zeta \; d^T y(\zeta)$$
$$\text{s.t.} \quad A_\zeta x + B_\zeta y(\zeta) \le b_\zeta \quad \forall \zeta \in Z$$

$$\min \quad c^T x + \eta$$
$$\text{s.t.} \quad \eta \ge d^T y(\zeta) \quad \forall \zeta \in Z$$
$$A_\zeta x + B_\zeta y(\zeta) \le b_\zeta \quad \forall \zeta \in Z$$

- **Instead of Z, use a subset**
- **Obtain a lower bound**
- **Idea: gradually add scenarios to obtain better lower bounds**

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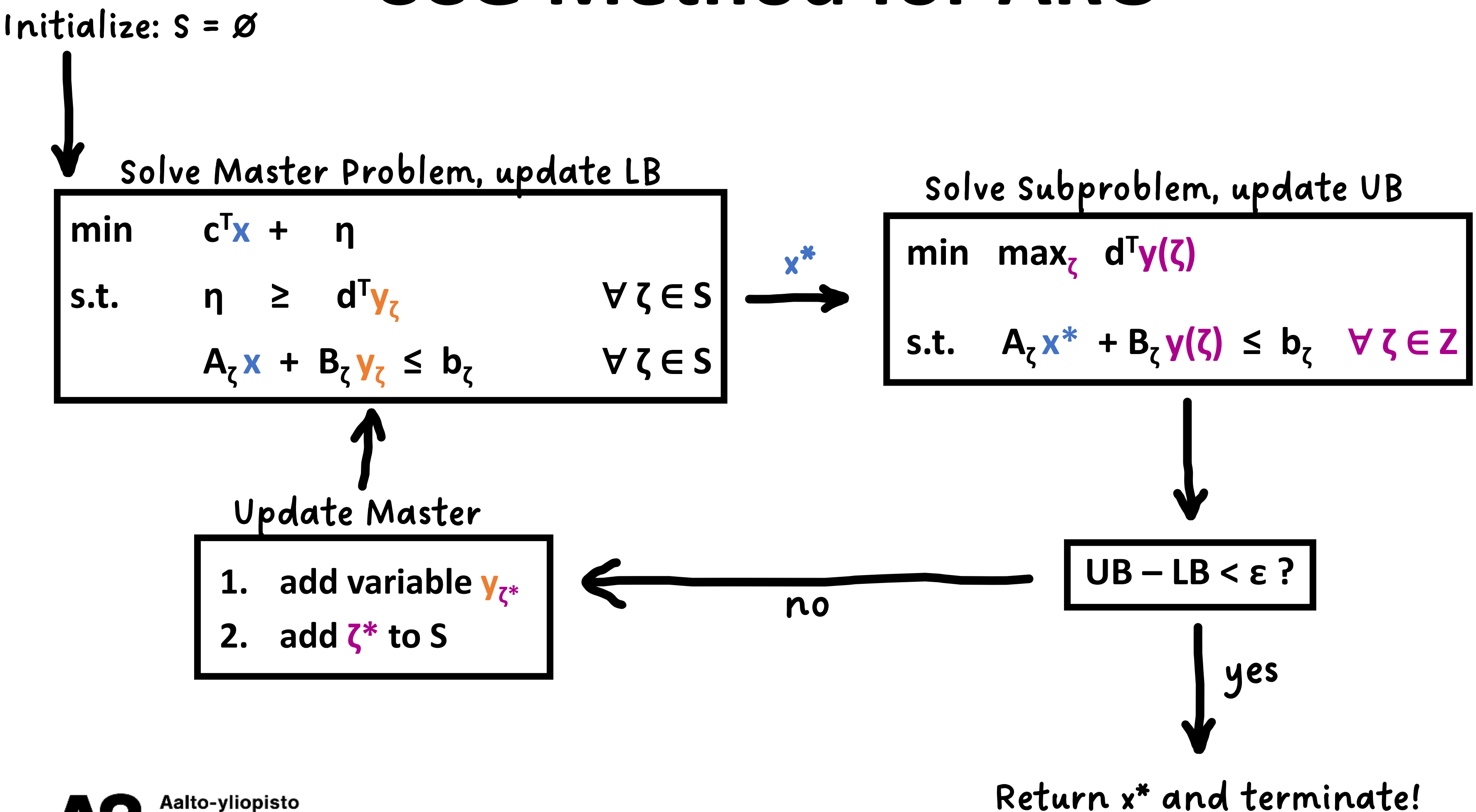
CCG Method for ARO



Example

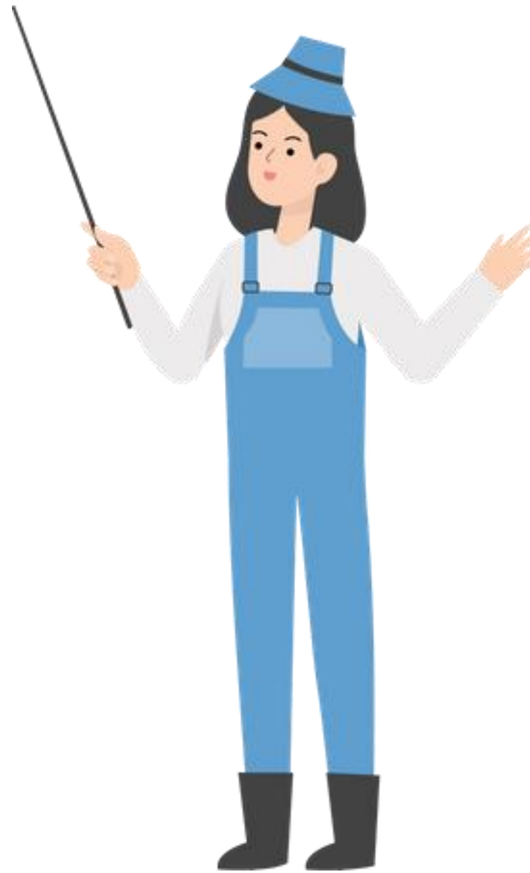
$$\min \quad x$$

$$\text{s.t.} \quad y_1(\zeta) + y_2(\zeta) \leq 50x \qquad \forall \zeta \in [0, 50]$$

$$y_1(\zeta) \geq 100 + \zeta \qquad \forall \zeta \in [0, 50]$$

$$y_2(\zeta) \geq 150 - \zeta \qquad \forall \zeta \in [0, 50]$$

$$x, y, s \geq 0$$



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Example

LB = 0

$$\min \quad x \qquad \text{s.t.} \quad x \geq 0$$

$x^* = 0$ →

UB = 1000·250 = 250 000 for $\zeta^* = 0$

$$\begin{aligned}
\min \quad & \max_\zeta \; 1000\,(s_1(\zeta) + s_2(\zeta)) \\
\text{s.t.} \quad & y_1(\zeta) + y_2(\zeta) \leq 50 \cdot 0 && \forall \zeta \in [0, 50] \\
& y_1(\zeta) + s_1(\zeta) \geq 100 + \zeta && \forall \zeta \in [0, 50] \\
& y_2(\zeta) + s_2(\zeta) \geq 150 - \zeta && \forall \zeta \in [0, 50] \\
& y, s \geq 0
\end{aligned}$$

↓

UB – LB < ε ?

← no

1. add variables y^0, s^0
2. add 0 to S

Example

LB = 5

$$\begin{aligned} \text{Min} \quad & x + \eta \\ \text{s.t.} \quad & \eta \geq 1000\,(s^0_1 + s^0_2) \\ & y^0_1 + y^0_2 \leq 50x \\ & y^0_1 + s^0_1 \geq 100 + 0 \\ & y^0_2 + s^0_2 \geq 150 - 0 \\ & x, y, s \geq 0 \end{aligned}$$

$x^* = 5$ →

UB = 5

$$\begin{aligned} \min \quad & \max_\zeta \; 1000\,(s_1(\zeta) + s_2(\zeta)) \\ \text{s.t.} \quad & y_1(\zeta) + y_2(\zeta) \leq 50 \cdot 5 && \forall \zeta \in [0, 50] \\ & y_1(\zeta) + s_1(\zeta) \geq 100 + \zeta && \forall \zeta \in [0, 50] \\ & y_2(\zeta) + s_2(\zeta) \geq 150 - \zeta && \forall \zeta \in [0, 50] \\ & y, s \geq 0 \end{aligned}$$

1. add variables y^0, s^0
2. add 0 to S

UB – LB < ε ?

yes

Reserve land for 5 trees!

Remarks

- Tractability depends on uncertainty set Z and subproblem complexity
- Convergence guaranteed for discrete and polyhedral Z
- Accommodates integer recourse
- Works well if the number of relevant scenarios is small
- Requires complete recourse

Conclusion

Conclusion

- **ARO rocks!**
- **Don’t need probability distributions**
- **Can model risk-averse behaviour**
- **Tractability depends strongly on exact circumstances**
- **Other approaches:**
    - duality-based reformulations
    - benders-style decompositions
    - piecewise linear recourse functions

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References

Ben-Tal, A., Goryashko, A., Guslitzer, E. *et al.* Adjustable robust solutions of uncertain linear programs . *Math. Program., Ser. A* **99**, 351–376 (2004).

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Performance of CCG vs Benders

Table 2
Performance of Benders-dual and C&CG algorithms on 30 × 30 instances.

Γ	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%	Avg.
BD (CPU sec.)	22.71	24.27	25.14	23.76	22.98	24.55	25.29	25.61	25.07	22.49	24.19
C&CG (CPU sec.)	1.35	2.59	3.12	2.54	1.85	2.51	1.90	2.17	1.39	0.38	1.98
Ratio	**16.82**	**9.37**	**8.06**	**9.35**	**12.42**	**9.78**	**13.31**	**11.80**	**18.04**	**59.18**	**16.81**
BD (# iter.)	65.4	59.4	56.8	50	47.6	45.6	45.8	43.7	43.3	42.1	49.97
C&CG (# iter.)	4.2	5.8	6.5	5.3	5.1	5.7	4.6	5.4	4	2	4.86
Ratio	**15.57**	**10.24**	**8.74**	**9.43**	**9.33**	**8.00**	**9.96**	**8.09**	**10.83**	**21.05**	**11.12**
BD Master (sec./iter.)	0.14	0.13	0.12	0.11	0.11	0.11	0.11	0.11	0.11	0.10	0.12
C&CG master (sec./iter.)	0.12	0.15	0.16	0.14	0.12	0.14	0.12	0.13	0.11	0.07	0.13
Ratio	**1.17**	**0.87**	**0.75**	**0.79**	**0.92**	**0.79**	**0.92**	**0.85**	**1.00**	**1.43**	**0.95**

Table 3
Performance of Benders-dual and C&CG algorithms on 70 × 70 instances.

Γ	10%	20%	30%	40%	50%	60%	70%	80%	90%	100%	Avg.
BD (CPU sec.)	776.42	1580.71	1367.34	1300.44	1002.96	935.42	672.68	735.81	619.7	466.68	945.82
C&CG (CPU sec.)	26.16	21.27	72.3	65.22	37.88	54.62	16.72	17.64	9.66	1.55	32.3
Ratio	**29.68**	**74.32**	**18.91**	**19.94**	**26.48**	**17.13**	**40.23**	**41.71**	**64.15**	**301.08**	**63.36**
BD (# iter.)	203.9	152.1	117.5	127.1	137.4	143.6	126.3	134.2	136.6	132.4	141.11
C&CG (# iter.)	6.8	5	4.9	5	5.2	5.9	4.5	5.1	4.9	2	4.93
Ratio	**29.99**	**30.42**	**23.98**	**25.42**	**26.42**	**24.34**	**28.07**	**26.31**	**27.88**	**66.20**	**30.90**
BD Master (sec./iter.)	1.13	0.79	0.57	0.56	0.46	0.41	0.34	0.35	0.33	0.3	0.52
C&CG Master (sec./iter.)	1.45	0.58	0.57	0.58	0.55	0.72	0.47	0.5	0.51	0.12	0.61
Ratio	**0.78**	**1.36**	**1.00**	**0.97**	**0.84**	**0.57**	**0.72**	**0.70**	**0.65**	**2.50**	**1.01**