Multistage stochastic programming

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Outline

Introduction

$$
\begin{array}{llllllllll}
\text{Min} & c_1^T x_1 & + & c_2^T x_2 & + & c_3^T x_3 & + & \dots & + & c_T^T x_T \\
\text{s.t.} & A_{11} x_1 & & & & & & & & = b_1, \\
 & A_{21} x_1 & + & A_{22} x_2 & & & & & & = b_2, \\
 & & & A_{32} x_2 & + & A_{33} x_3 & & & & = b_3, \\
 & \multicolumn{9}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\
 & & & & & & & A_{T,T-1} x_{T-1} & + & A_{TT} x_T = b_T, \\
 & x_1 \geq 0, & & x_2 \geq 0, & & x_3 \geq 0, & & \dots & & x_T \geq 0.
\end{array}
\tag{3.1}
$$

Figure: Ruszczyński & Shapiro (2003)

- $t \in \{1, 2, \dots, T\} = \mathcal{T}$ stages
- $x_t \in \mathbb{R}_+^n$ decision variables
- $c_t, b_t, A_{t_1 t_2}$ random

Example 1, Inventory management

(Example from Shapiro & Philpott (2007))

- A company sells a product depending on random demand D_t.
- Each stage $t \in \mathcal{T}$ current inventory I_t is observed and replenished to S_t ($S_t \geq I_t$).
- When $S_t > I_t$ new products are ordered $(S_t - I_t)$ for c_t per unit. These orders cost $c_t(S_t - I_t)$
- Demand is unfulfilled when $(D_t > S_t)$ and incurs costs b_t per unit. The cost for unfulfilled demand is $b_t(D_t - S_t)$
- Holding costs h_t per unit. The holding cost is $h_t(S_t - D_t)$

The total costs (to minimize) are
$G(I, S, D) = c_t(S_t - I_t) + [b_t(D_t - S_t)]_+ + [h_t(S_t - D_t)]_+$
$([a]_+ = \max(0, a))$

Example 1

The task is to minimize the expectation (or some risk measure) of $G(I, S, D) = c_t(S_t - I_t) + [b_t(D_t - S_t)]_+ + [h_t(S_t - D_t)]_+$.

$$\min_{S_t} \mathbb{E}[G(I, S, D)], \tag{1}$$

$$\text{st.} I_{t+1} = S_t - D_t, \tag{2}$$

$$0 \leq I_t \leq S_t, \tag{3}$$

$$t \in \{1, 2, ..., T\} \tag{4}$$

Information Structure

At first we make a (first stage) decision without any observations.
Then we observe a realization of the random variable(s). Then a decision is made and the next observation is made.
Therefore it makes sense to study optimal decision making backwards.
Let $I_{[t]} = (I_1, ..., I_t)$ be observations of inventory until t (and $D_{[t]} = (D_1, ..., D_t)$ and so on).
Suppose we are at the final stage, we know $I_{[T]}$ and $S_{[T-1]}$. The problem to solve is:

$$\min_{S_T \geq I_T} c_T(S_T - I_T) + \mathbb{E}[[b_T(D_T - S_T)]_+ + [h_T(S_T - D_T)]_+ | D_{[T-1]}]. \quad (5)$$

Let $V_T(I_T, D_{[T-1]})$ be the optimal value of this problem. V is also called the cost to go term
(Note also that the expectation is conditional).

Dynamic programming relation

$$V_T(I_T, D_{[T-1]}) = \min_{S_T \geq I_T} c_T(S_T - I_T) + \tag{6}$$

$$\mathbb{E}[[b_T[(D_T - S_T)]_+ + [h_T(S_T - D_T)]_+ | D_{[T-1]}]. \tag{7}$$

Similarly for S_{T-1} we have that (but with the expectation of the previous V_T)

$$V_{T-1}(I_{T-1}, D_{[T-2]}) = \min_{S_{T-1} \geq I_{T-1}} c_{T-1}(S_{T-1} - I_{T-1}) + \tag{8}$$

$$\mathbb{E}[V_T(S_T - D_T, D_{[T-1]}) + \tag{9}$$

$$[b_{T-1}(D_{T-1} - S_{T-1})]_+ + \tag{10}$$

$$[h_{T-1}(S_{T-1} - D_{T-1})]_+ | D_{[T-2]}]. \tag{11}$$

For any $t > 1$:

$$V_t(I_t, D_{[t-1]}) = \min_{S_t \geq I_t} c_t(S_t - I_t) + \tag{12}$$

$$\mathbb{E}[V_t(S_t - D_t, D_{[t-1]}) + [b_t(D_t - S_t)]_+ + [h_t(S_t - D_t)]_+ | D_{[t-1]}]. \tag{13}$$

Dynamic programming relation

For any $t > 1$:

$$V_t(I_t, D_{[t-1]}) = \min_{S_t \geq I_t} c_t(S_t - I_t) + \quad (14)$$

$$\mathbb{E}[V_t(S_t - D_t, D_{[t-1]}) + [b_t(D_t - S_t)]_+ + [h_t(S_t - D_t)]_+ | D_{[t-1]}]. \quad (15)$$

and at the first stage:

$$V_1(I_1) = \min_{S_1 \geq I_1} c_1(S_1 - I_1) + \quad (16)$$

$$\mathbb{E}[V_2(S_1 - D_1, D_1)[b_1(D_1 - S_1)]_+ + [h_1(S_1 - D_1)]_+]. \quad (17)$$

Conditional expectations are tricky to deal with.
These can be simplified with a mutual independence assumption.

General formulation

- Probability space $\Omega, \mathcal{F}, \Pr$
- Sequence of random variables $\epsilon_t : \Omega \to \mathbb{R}^{d_t}, t \in \mathcal{T}$
- Decisions $x_t \in \mathbb{R}^{n_t}$
- Data and decisions up to t : $x_{[t]} = (x_1, ..., x_t), \epsilon_{[t]} = (\epsilon_1, ..., \epsilon_t)$

Decisions are followed by observations, followed by decisions and so on.
Values of x_t depend on $\epsilon_{[t]}$ (information up to t but not after).
Consider x_t as a random process, $x_t : \Omega \to \mathbb{R}^{n_t}$.
Non-anticipativity constraints make sure no decision is made with future information (when finding solutions):

$$x_t = \mathbb{E}[x_t | \epsilon_{[t]}] \tag{18}$$

Measurable wrt. the sigma algebra generated by $\epsilon_{[t]}$

General formulation

Objective function $F : \mathbb{R}^{\sum n_t \times \sum d_t} \to \mathbb{R}$
Constraints $G_t^i : \mathbb{R}^{\sum n_t \times \sum d_t} \to \mathbb{R}, i \in \{1, ..., m_t\}$.

The problem is:

$$\min_x \mathbb{E}[F(x_{[T]}(\omega), \epsilon_{[T]}(\omega))], \tag{19}$$

$$\text{st. } G_t^i(x_{[t]}(\omega), \epsilon_{[t]}(\omega)) \leq 0, \tag{20}$$

$$x_t \in X_t. \tag{21}$$

(Measure theoretic expectation: Pr. space $(\Omega, \mathcal{F}, \Pr)$, random variables $Y, Y' : \Omega \to \mathbb{R}$ are $\mathcal{F}$ measurable.
$(Y')^{-1}[A \subseteq \mathbb{R}] = \{\omega \in \Omega : Y'(\omega) \in A\} = \mathcal{A}$
$\mathbb{E}(Y) = \int_\Omega Y(\omega) d\Pr(\omega)$,
$\mathbb{E}(Y | Y' \in A) = \int_{\mathcal{A}} Y(\omega) d\Pr(\omega))$

General formulation

$$\min_x \mathbb{E}[F(x_{[T]}(\omega), \epsilon_{[T]}(\omega))], \tag{22}$$

$$\text{st. } G_t^i(x_{[t]}(\omega), \epsilon_{[t]}(\omega)) \leq 0, \tag{23}$$

$$x_t \in X_t. \tag{24}$$

Again it makes sense to study the problem backwards:
at $t = T$ we solve:

$$V_T(x_{[T-1]}, \epsilon_{[T]}) = \min_{x_T} F(x_{[T-1]}, x_T, \epsilon_{[T]}), \tag{25}$$

$$\text{st. } G_T^i(x_{[T-1]}, x_T, \epsilon_{[T]}) \leq 0, \tag{26}$$

$$x_T \in X_T. \tag{27}$$

General formulation

At $t = T-1$:

$$V_{T-1}(x_{[T-2]}, \epsilon_{[T-1]}) = \min_{x_{T-1}} \mathbb{E}[V_T(x_{[T-1]}, \epsilon_{[T]})|\epsilon_{[T-1]}], \quad (28)$$

$$\text{st.} \;\; G^i_{T-1}(x_{[T-2]}, x_{T-1}, \epsilon_{[T-1]}) \leq 0, \quad (29)$$

$$x_{T-1} \in X_{T-1}. \quad (30)$$

And so on $t < T$:

$$V_t(x_{[t-1]}, \epsilon_{[t]}) = \min_{x_t} \mathbb{E}[V_{t+1}(x_{[t]}, \epsilon_{[t+1]})|\epsilon_{[t]}], \quad (31)$$

$$\text{st.} \;\; G^i_t(x_{[t-1]}, x_t, \epsilon_{[t]}) \leq 0, \quad (32)$$

$$x_t \in X_t. \quad (33)$$

Linear case

$$
\begin{array}{llllllll}
\text{Min} & c_1^T x_1 & + & c_2^T x_2 & + & c_3^T x_3 & + & \dots & + & c_T^T x_T & & \\
\text{s.t.} & A_{11} x_1 & & & & & & & & & = & b_1, \\
& A_{21} x_1 & + & A_{22} x_2 & & & & & & & = & b_2, \\
& & & A_{32} x_2 & + & A_{33} x_3 & & & & & = & b_3, \\
& \multicolumn{11}{l}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\
& & & & & & & A_{T,T-1} x_{T-1} & + & A_{TT} x_T & = & b_T, \\
& x_1 \geq 0, & & x_2 \geq 0, & & x_3 \geq 0, & & \dots & & x_T \geq 0. & &
\end{array}
\tag{3.1}
$$

Figure: Ruszczyński & Shapiro (2003)

Stair-case formulation

$\epsilon_t = [A_{t-1,t}, A_{t,t}, b_t, c_t]$.
ϵ_1 is known and not random.
x_1 is made before any realization of randomness.

Lower triangular formulation

$$
\begin{array}{lllllllllll}
\text{Min} & c_1^T x_1 & + & c_2^T x_2 & + & c_3^T x_3 & + & \dots & + & c_T^T x_T & \\
\text{s.t.} & A_{11} x_1 & & & & & & & & & = b_1, \\
 & A_{21} x_1 & + & A_{22} x_2 & & & & & & & = b_2, \\
 & A_{31} x_1 & + & A_{32} x_2 & + & A_{33} x_3 & & & & & = b_3, \\
 & \cdots\cdots & & & & & & & & & \\
 & A_{T1} x_1 & + & A_{T2} x_2 & + & \dots & + & A_{T,T-1} x_{T-1} & + & A_{TT} x_T & = b_T, \\
 & x_1 \geq 0, & & x_2 \geq 0, & & x_3 \geq 0, & & \dots & & x_T \geq 0. &
\end{array}
\tag{3.6}
$$

Figure: Ruszczyński & Shapiro (2003)

Can be converted to stair-case formulation using state variables and equations: $[A_{t1}, A_{t2}, \dots, A_{t,t-1}] x_{[t-1]} + A_{t,t} x_t = b_t$

Linear case, subproblems

At $t = T$:

$$V_T(x_{T-1}, \epsilon_{[T]}) = \min_{x_T} c_T^* x_T, \tag{34}$$

$$\text{st. } A_{T-1,T} x_{T-1} + A_{T,T} x_T = b_T. \tag{35}$$

$$V_t(x_{t-1}, \epsilon_{[t]}) = \min_{x_t} c_t^* x_t + \mathbb{E}[V_{t+1}(x_t, \epsilon_{[t+1]}) | \epsilon_{[t]}], \tag{36}$$

$$\text{st. } A_{t-1,t} x_{t-1} + A_{t,t} x_t = b_t. \tag{37}$$

At $t = 1$:

$$\min_{x_1} c_1^* x_1 + \mathbb{E}[V_2(x_1, \epsilon_2)], \tag{38}$$

$$\text{st. } A_{11} x_1 = b_1. \tag{39}$$

A modest assumption of mutual stagewise independence of the random variables gets rid of the conditional expectations.

Finitely many scenarios

- K scenarios.
- First a first stage decision is made without any data
- Then a sequence of observation $\rightarrow$ decision $\rightarrow$ observation$\rightarrow$... is repeated
- In other words decisions are made depending on realization of random variables
- $\mathbb{E}[x] = \sum_k p^k x^k$.

It is important to structure the problem such that decisions(solutions) aren't made with future information.

Two stage relaxation

$$
\begin{array}{llllll}
\text{Min} \sum_{k=1}^{K} p_k[(c_1)^T x_1^k + (c_2^k)^T x_2^k + (c_3^k)^T x_3^k + & \cdots & + (c_T^k)^T x_T^k] \\
\text{s.t.} \quad A_{11} x_1^k & & & = b_1, \\
A_{21}^k x_1^k + A_{22}^k x_2^k & & & = b_2^k, \\
\quad A_{32}^k x_2^k + A_{33}^k x_3^k & & & = b_3^k, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& & A_{T,T-1}^k x_{T-1}^k + A_{TT}^k x_T^k & = b_T^k, \\
x_1^k \geq 0, \quad x_2^k \geq 0, \quad x_3^k \geq 0, & \ldots & x_T^k \geq 0, \\
k=1,\ldots,K. & & & (3.8)
\end{array}
$$

Figure: Ruszczyński & Shapiro (2003)

and only $x_1^j = x_1^k, \forall j \neq k$, can be understood as a two stage SLP (with a lot more scenarios) which is a relaxation of the MSSLP. $T = A_{21}$ and W is

$$
\begin{bmatrix}
A_{22} & 0 & \cdots\cdots & 0 & 0 \\
A_{32} & A_{33} & \cdots\cdots & 0 & 0 \\
 & & \cdots\cdots\cdots\cdots & & \\
0 & 0 & \cdots\cdots & A_{T,T-1} & A_{TT}
\end{bmatrix}.
$$

Non-anticipativity constraints in MSSLP with finitely many scenarios

$x_1^j = x_1^k, \forall j \neq k$

$x_t^k = x_t^j \forall k, j$ where $\epsilon_{[t]}^k = \epsilon_{[t]}^j \forall t$
This allows to frame the problem (with the constraints above!):

$$
\begin{array}{llr}
\text{Min} \sum_{k=1}^{K} p_k[(c_1)^T x_1^k + (c_2^k)^T x_2^k + (c_3^k)^T x_3^k + \quad \cdots \quad + (c_T^k)^T x_T^k] & & \\
\text{s.t.} \quad A_{11} x_1^k & = b_1, & \\
A_{21}^k x_1^k + A_{22}^k x_2^k & = b_2^k, & \\
A_{32}^k x_2^k + A_{33}^k x_3^k & = b_3^k, & \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & & \\
A_{T,T-1}^k x_{T-1}^k + A_{TT}^k x_T^k & = b_T^k, & \\
x_1^k \geq 0, \quad x_2^k \geq 0, \quad x_3^k \geq 0, \quad \ldots \quad x_T^k \geq 0, & & \\
k = 1, \ldots, K. & & (3.8)
\end{array}
$$

Figure: Ruszczyński & Shapiro (2003)

Example 2, Scenario trees

Scenario trees provide vizualisation of possible realizations of ϵ_t. In this example, only b_t are random. Each node corresponds to a realization, arcs are conditional probabilities of each realization.

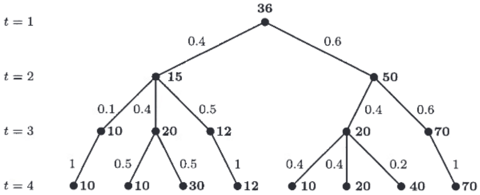

Figure: Ruszczyński & Shapiro (2003)

Suppose the problem is (univariate everything)

$$\min_{x_t \geq 0} x_t + \mathbb{E}[V_{t+1}(x_t, b_{t+1})|b_t] \tag{40}$$

$$x_{t-1} + x_t = b_t \tag{41}$$

Example 2

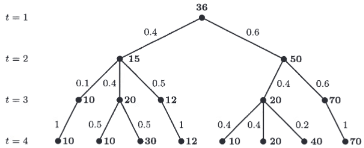

Figure: Ruszczyński & Shapiro (2003)

The final cost to go is: $V_4(x_4, b_4) = \min_{x_4} x_4, x_3 + x_4 = b_4$
We see that the minimum is either $V_4 = b_4 - x_3$, if $x_3 \leq b_4$ or infeasible $(x \geq 0)$.
Next stage:

$$\min_{x_3 \geq 0} x_3 + \mathbb{E}[b_4 - x_3 | b_3 = 20] \tag{42}$$

$$x_2 + x_3 = 20 \tag{43}$$

Scenario trees

More generally we can think of end nodes in the scenario tree as all possible scenarios. Sometimes the scenario probabilities are found using conditional probabilities (arcs) and sometimes the conditional probabilities are found using probabilities at different nodes each stage.

Using scenario trees non-anticipativity constraints can be expressed as:

$$x_t^k = \frac{\sum_{i \in A(k)_t} p^i x_t^i}{\sum_{i \in A(k)_t} p^i}. \tag{44}$$

Where $A(k)_t$ are scenarios with shared history until t

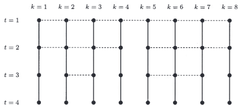

Figure: Ruszczyński & Shapiro (2003)

References

Ruszczyński, A. & Shapiro, A. (2003), 'Stochastic programming models', *Handbooks in operations research and management science* **10**, 1–64.

Shapiro, A. & Philpott, A. (2007), 'A tutorial on stochastic programming', *Manuscript. Available at www2. isye. gatech. edu/ashapiro/publications. html* **17**.