Risk Measures

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Outline

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II. **Techniques** – ***How to model risk?***
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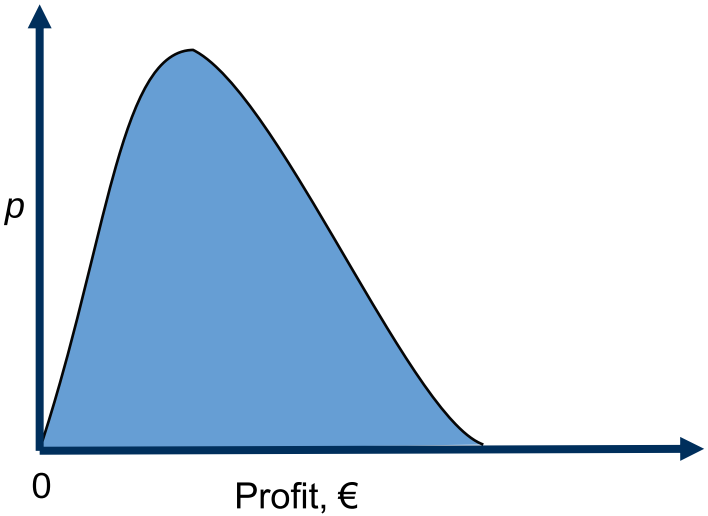
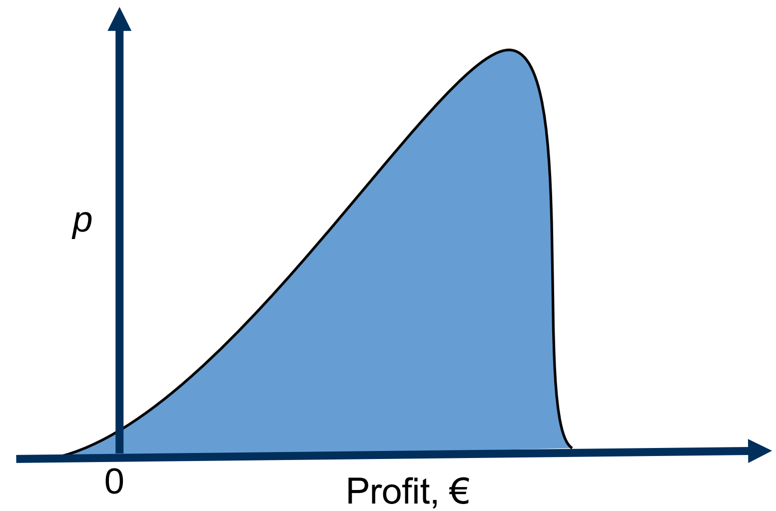
V. **References** – ***Where is in-depth knowledge to be found?***

I. Introduction

Investment opportunities – which one would you choose?



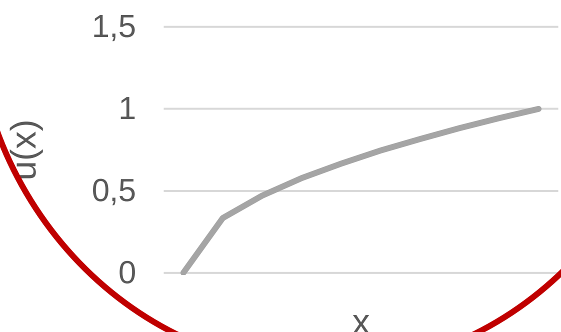


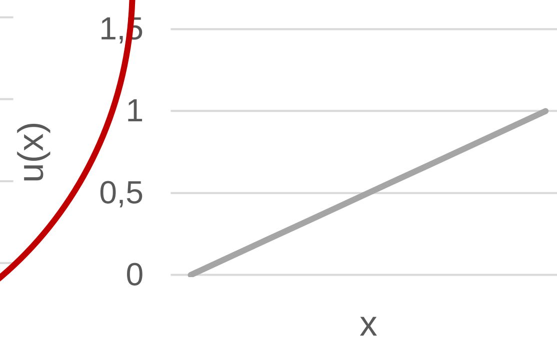
Reminder: Risk attitudes

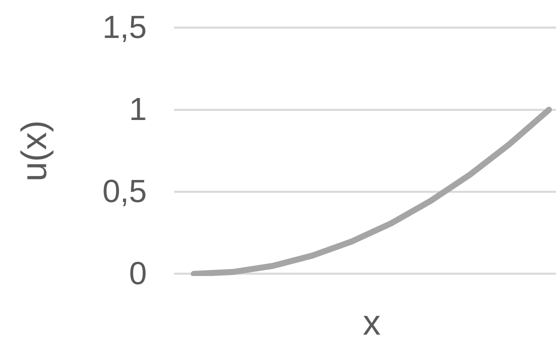
Risk averse

Risk neutral

Risk seeking







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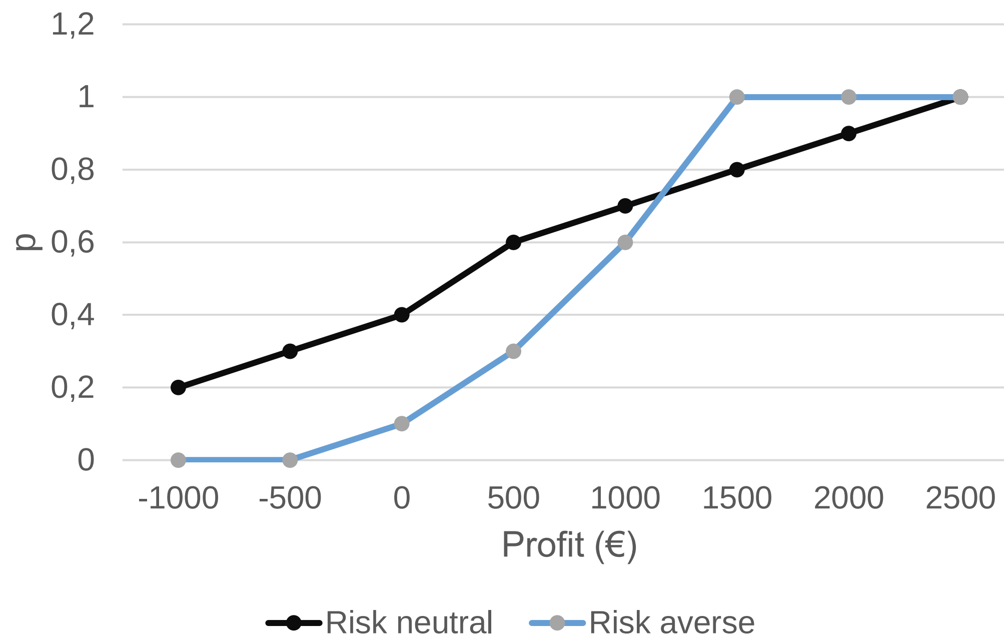
Why should stochastic optimisation be risk averse?

$$c^T x + \sum_{\omega \in \Omega} \pi(\omega) q(\omega)^T y(\omega)$$



II. Techniques

Risk-neutral problem

Max. $\boldsymbol{x}, \boldsymbol{y}(\boldsymbol{\omega})$

$$\boldsymbol{c}^T\boldsymbol{x} + \sum_{\boldsymbol{\omega}\in\boldsymbol{\Omega}} \boldsymbol{\pi}(\boldsymbol{\omega})\boldsymbol{q}(\boldsymbol{\omega})^T\boldsymbol{y}(\boldsymbol{\omega})$$

s. t. $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$

$$\boldsymbol{T}(\boldsymbol{\omega})\boldsymbol{x} + \boldsymbol{W}(\boldsymbol{\omega})\boldsymbol{y}(\boldsymbol{\omega}) = \boldsymbol{h}(\boldsymbol{\omega}) \forall \boldsymbol{\omega} \in \boldsymbol{\Omega}$$

$$\boldsymbol{x} \in \boldsymbol{X}, \boldsymbol{y}(\boldsymbol{\omega}) \in \Upsilon, \boldsymbol{\omega} \in \boldsymbol{\Omega}$$

~~Risk-neutral~~ problem

Max. $x, y(\omega)$

$$(1-\beta)\left(c^T x + \sum_{\omega \in \Omega} \pi(\omega) q(\omega)^T y(\omega)\right) - \beta \cdot risk$$

s. t. $Ax = b$

$$T(\omega)x + W(\omega)y(\omega) = h(\omega) \forall \omega \in \Omega$$

$$risk \leq \delta$$

$$x \in X, y(\omega) \in \Upsilon, \omega \in \Omega$$

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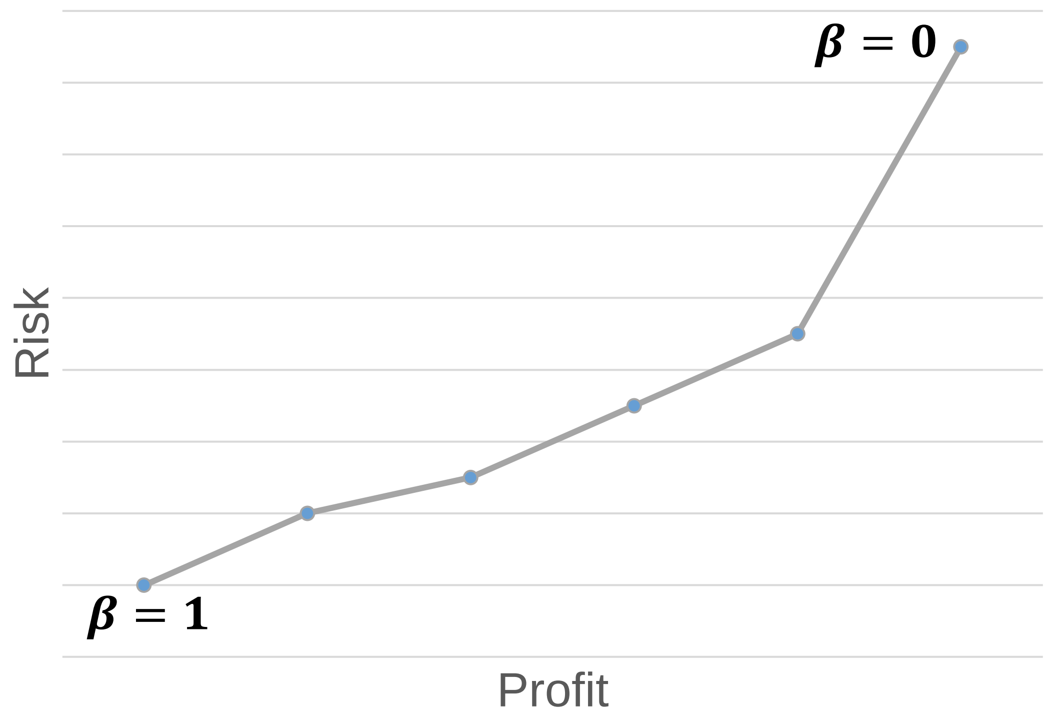
Risk vs. profit

Efficient point

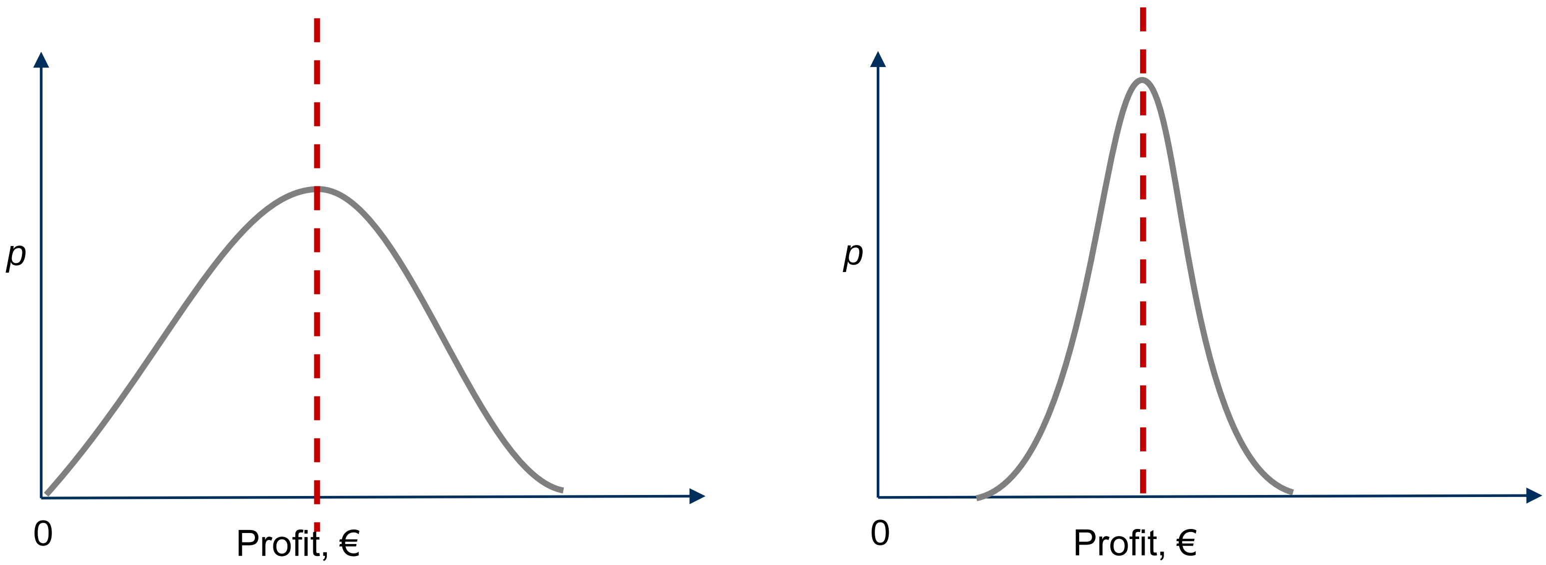
Efficient frontier



Risk measures

- Variance
- Shortfall probability
- Expected shortage
- Value-at-Risk VaR
- Conditional Value-at-Risk CVaR

Variance as risk measure



Minimising variance

$$\textbf{Max. } x, y(\omega)$$

$$(1-\beta)\left(c^T x + \sum_{\omega\in\Omega} \pi(\omega) q(\omega)^T y(\omega)\right)$$

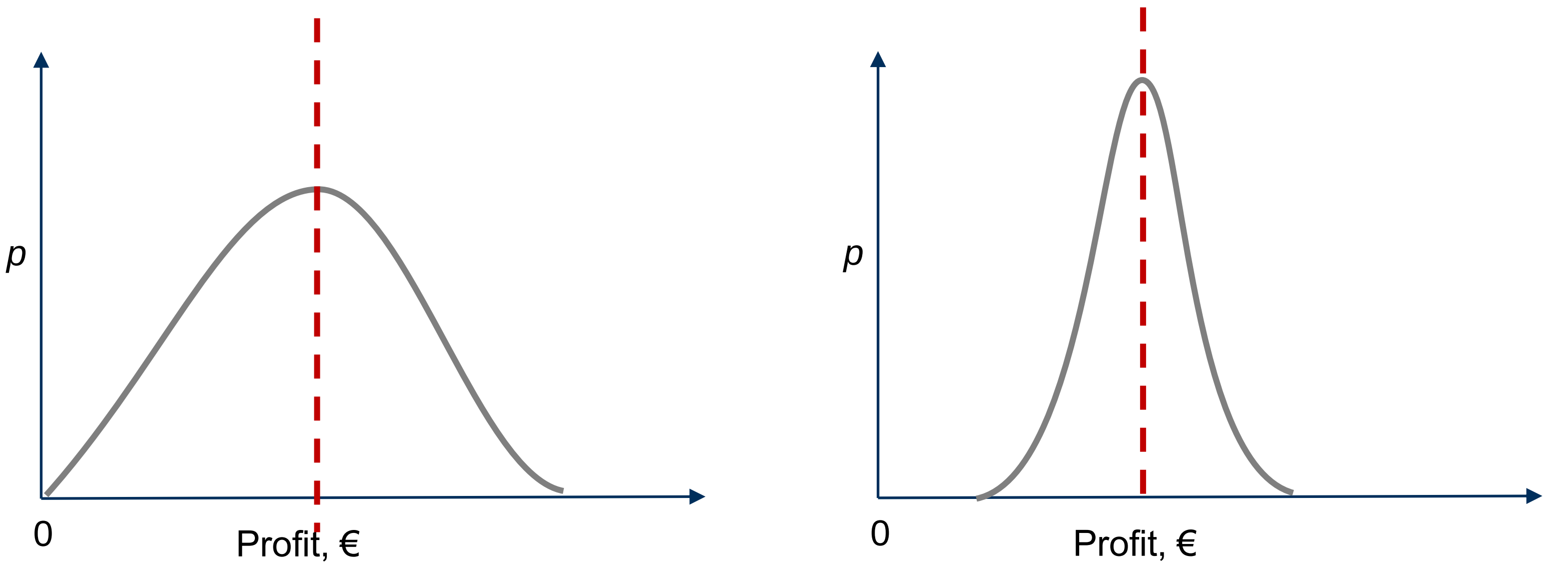
$$-\beta \sum_{\omega\in\Omega} \pi(\omega) \left(\underbrace{f(x,\omega)}_{\text{profit}} - \sum_{\omega'\in\Omega} \pi(\omega') (f(x,\omega') \right)^2$$

$$\text{s. t. } Ax = b$$

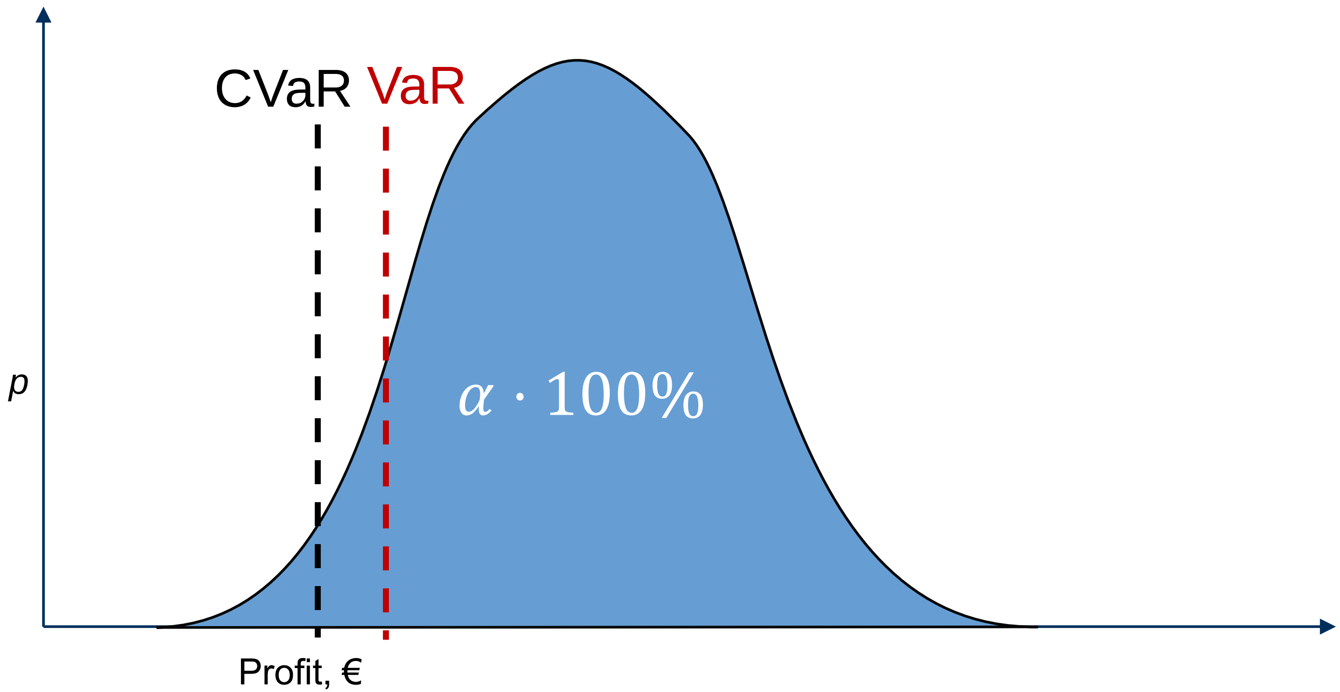
$$T(\omega)x + W(\omega)y(\omega) = h(\omega) \forall \omega \in \Omega$$

$$x \in X, y(\omega) \in \Upsilon, \omega \in \Omega$$

Variance as risk measure



(Conditional) Value-at-Risk



Maximising CVaR (conditional Value-at-Risk)

Max. $x, y(\omega), \eta, s(\omega)$

$$(1-\beta)\left(c^T x + \sum_{\omega\in\Omega} \pi(\omega) q(\omega)^T y(\omega)\right)$$

$$-\beta\left(\eta - \frac{1}{1-\alpha}\sum_{\omega\in\Omega} \pi(\omega) s(\omega)\right)$$

s. t. $Ax = b$

$$T(\omega)x + W(\omega)y(\omega) = h(\omega) \forall \omega \in \Omega$$

$$\eta - \left(c^T x + q(\omega)^T y(\omega)\right) \leq s(\omega) \forall \omega \in \Omega$$

$$s(\omega) \geq 0 \forall \omega \in \Omega$$

$$x \in X, y(\omega) \in Y, \omega \in \Omega$$

CVaR as risk measure

- **Widely used**
- **Coherent**
- **Linear expression** (Rockafellar & Uryasev):

$$\textbf{Max. } \boldsymbol{\eta} + \frac{1}{q(1-\alpha)} \sum_{k=1}^{q} \boldsymbol{u_k}$$

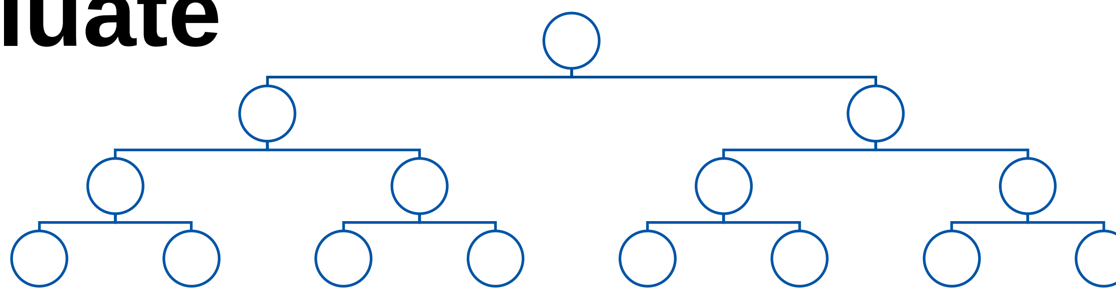
$$\textbf{s.t. } \boldsymbol{x_j \geq 0 \forall j, \sum_{j=1}^{n} x_j = 1}$$

$$\boldsymbol{\mu(x) \leq -R}$$

$$\boldsymbol{u_k \geq 0, x^T y_k + \eta + u_k \geq 0 \forall k}$$

Multi-stage stochastic programming – a harder task

- Weighting between stages
- Nested structure
- Where to evaluate



III. Example

Example: Electricity retailer by Conejo et al., 2010:

Contracts $f = 1,2,3$
Purchasing prices λ_f^F,
maximum quantities $X_f^{\max}$

Time periods $t = 1,2,3$
Demands P_t^C
Selling prices λ^C

Scenarios $\omega = 1,2,\ldots,10$
Pool prices $\lambda_{t\omega}^P$

Example: Electricity retailer by Conejo et al., 2010:

$$\text{Max.}\ x_f, y_{t\omega}, s_\omega, \eta$$

$$(1-\beta)\left(\sum_{t=1}^{3}\lambda^C P_t^C - \sum_{\mathrm{f}=1}^{3}\ \sum_{\mathrm{t}=1}^{3}\lambda_f^F x_f - \sum_{\omega=1}^{10}\pi_\omega \sum_{t=1}^{3}\lambda_{t\omega}^P y_{t\omega}\right)$$

$$+\beta\left(\eta - \frac{1}{1-\alpha}\sum_{\omega=1}^{10}\pi_\omega s_\omega\right)$$

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Example: Electricity retailer by Conejo et al., 2010:

s. t. $0 \le x_f \le X_f^{max}, f = 1, 2, 3$

$$\sum_{f=1}^{3} x_f + y_{t\omega} = P_t^C \; \forall t \forall \omega$$

$$y_{t\omega} \ge 0 \forall t \forall \omega$$

$$\eta - \left(\sum_{t=1}^{3} \lambda^C P_t^C - \sum_{f=1}^{3} \sum_{t=1}^{3} \lambda_f^F x_f - \sum_{t=1}^{3} \lambda_{t\omega}^P y_{t\omega} \right) \le s_\omega \forall \omega$$

Example: Electricity retailer by Conejo et al., 2010:

	$\beta = 0$	$\beta = 1$
Optimal decision x*	(0, 0, 0)	(50,30,25)
CVaR(0.8)($)	-801.25	-343.00
Expected profit ($)	618.75	208.65

IV. Conclusion

Summary

- **Uncertainty → ∃ Risk**
- **High profit → High risk**
- **Modelling risk**
  - Conditional Value-at-Risk
  - Choice of parameters

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- Conejo et al.: *Decision Making under Uncertainty in Electricity Markets*. Springer Science+Business Media 2010.
- Homem-de-Mello and Pagnoncelli: *Risk Aversion in Multistage Stochastic Programming: A Modeling and Algorithmic Perspective*. European Journal of Operational Research 249 / 2016, pp. 188-199.
- Liesiö, Punkka, Salo, Vilkkumaa: *Decision Making and Problem Solving – Lecture 3*. Lecture slides, Aalto University 2021.
- Rockafellar and Uryasev: Optimization of conditional value-at-risk. Journal of risk 2 / 2000, pp. 21-42.