Math Camp - Concavity

Concavity

We've already seen examples of functions where the max isn't unique or a local max isn't a global max.

We also have seen problems where the critical points identify things other than a max or min

- If $f : \mathbb{R} \to \mathbb{R}$, if $f''(x) \leq 0$ everywhere, then clearly everything (but uniqueness) isn't a problem.
- The same is true if $D^2 f$ is negative semidefinite everywhere
- Turns out, this describes an important class of functions.

Convex Sets

We've been talking a lot about drawing lines, what sort of sets make this possible?

Definition (Convex Set)

We say a set $X \subseteq \mathbb{R}^m$ is convex if for any $x, y \in X$, $\lambda x + (1 - \lambda) y \in X$ for all $\lambda \in (0, 1)$.

We can build a convex set from any set X.

- The convex hull of X: The smallest convex set containing X.
- Equvalently the set of all convex combinations of points in X.

Concavity

Definition (Concave/convex function)

Let $X \subseteq \mathbb{R}^m$ convex and $f : X \to \mathbb{R}$.

- f is concave if $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$ for any $\lambda \in (0,1)$. If the inequality is strict for all $x \neq y$, then the function is strictly concave.
- f is convex if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ for any $\lambda \in (0,1)$. If the inequality is strict for all $x \neq y$, then the function is strictly convex.

Convex and concave functions “draw” the boundaries of convex sets. f is convex iff $\{(x,y) : y \geq f(x)\}$ (the epigraph) is convex.

Examples

- Linear functions are both convex and concave.
- $f(x) = x^2$ is strictly convex.
- $f(x) = \sqrt{x}$ is strictly concave.
- $f(x) = x^3$ is neither.
- $f(x, y) = \alpha \log x + \beta \log y$ is strictly concave for any $\alpha, \beta > 0$.
- $f(x, y) = \min\{x, y\}$ is concave

Some properties

- $f(x)$ is convex iff $-f(x)$ is concave.
- $\alpha f(x) + \beta g(x)$ is concave if $f(x)$ and $g(x)$ are concave and $\alpha, \beta \geq 0$.
- If $f : X \to \mathbb{R}^m$ is concave and X is open, then $f(\cdot)$ is continuous.
- A concave function is differentiable almost everywhere.
- If f, g are concave and f is non-decreasing, then so is $f \circ g$.
- The pointwise minimum of two concave functions is concave.
- $f : \mathbb{R} \to \mathbb{R}$ concave, $x_3 > x_2 > x_1$

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \geq \frac{f(x_3) - f(x_2)}{x_3 - x_2}$$

Who cares?

Theorem

Let X be a convex set and $f : X \to \mathbb{R}$ be concave. Then any local maximum is a global maximum. Moreover, if f is strictly concave then it has at most one maximum.

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Let X be a convex set and $f : X \to \mathbb{R}$ be concave. Then any local maximum is a global maximum. Moreover, if f is strictly concave then it has at most one maximum.

Proof:

- Suppose x is a local, but not global maximizer. Then exists a y s.t. $f(y) > f(x)$.
- Then for all $\lambda \in (0, 1)$

  $$\begin{aligned} f(\lambda x + (1-\lambda)y) &\geq \lambda f(x) + (1-\lambda)f(y) \\ &> \min\{f(x), f(y)\} \\ &\geq f(x) \end{aligned}$$

  which is a contradiction. If a strictly concave function has two global maxes, x,y then

  $$f(\lambda x + (1-\lambda)y) > \lambda f(x) + (1-\lambda)f(y)$$

  which is a contradiction.

Quasiconcavity

The key property we used was really

$$f(\lambda x + (1-\lambda)y) > \min\{f(x), f(y)\}$$

We say a function is strictly quasiconcave if it satisfies this property. Equivalently a function is quasiconcave iff

$$U_y = \{x \in X : f(x) \geq y\}$$

are convex for all y. These are called the upper contour sets.

- Any concave function is quasiconcave.
- Quasiconcave functions need not be concave or convex or even continuous.

Quasiconcavity

We can prove the following

Theorem

$f : X \to \mathbb{R}$, X convex and assume $f(\cdot)$ attains it's maximizer.

- *If f is quasiconcave then the set of maximizers is convex.*
- *If f is strictly quasiconcave then the maximizer is unique.*

Derivatives

Concavity is a sort of finicky global property of the a function. Fortunately, we can usually check something simpler

Theorem

Let $f : \mathbb{R}^m \to \mathbb{R}$ be a twice continuously differentiable function. Then the following are equivalent

- *f is concave*
- *$D^2 f$ is negative semi-definite for all x.*
- $f(y) - f(x) \leq \nabla f(x) \cdot (y - x)$

Similarly, a function is quasiconcave iff $f(y) \geq f(x) \Rightarrow \nabla f(x)(y - x) \geq 0$.

Optimization

So if we show:

- The objective function is concave
- The set we are maximizing over is convex.

Then any solution to the FOCs is a global maximum. If we can also show the objective is strictly quasiconcave, then the maximum is unique!

Separating Hyperplane

When can we draw a line between two sets?

- Take any $p \in \mathbb{R}^m$ and some $a \in \mathbb{R}$. The set

$$h(p, c) = \{y : p \cdot y = c\}$$

  is a hyperplane. In $\mathbb{R}^2$ this is a line, in $\mathbb{R}^3$ a plane, etc.
- Our goal is a result like the following: $A, B \subseteq \mathbb{R}^m$. There exists a hyperplane $h(p, c)$ s.t. for all $a \in A$, $b \in B$

$$p \cdot a \leq c \leq p \cdot b.$$

  so our line separates the two sets.

Separating Hyperplanes

There are a number of these theorems, including very important results in functional analysis and in linear programming. Here are two

Theorem (Separating Hyperplane theorem)

Let $A \neq \emptyset$ *be a closed, convex set in* $\mathbb{R}^m$ *and* $B \neq \emptyset$ *is a compact, convex set in* $\mathbb{R}^m$. *If* $E \cap D = \emptyset$ *then there exists a* p *and an* d *s.t. for all* $a \in A$, $b \in B$, $p \cdot a < d < p \cdot b$

Theorem (Supporting Hyperplane theorem)

Let $A \neq \emptyset$ *be a convex set in* $\mathbb{R}^m$ *and* $x \in \mathbb{R}^m \setminus int(D)$. *Then exists a* p *and* c *s.t. for all* $a \in A$, $p \cdot a \leq c \leq p \cdot x$.

A good exercise is to think about what could go wrong if you relaxed any assumption.