Harmonic oscillator:

Consider a 1-D harmonic oscillator with the Hamiltonian

$$\hat{H} = \hat{T} + \hat{V}$$

$$= \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega_0^2 \hat{x}^2$$

$$\overset{\text{real-}}{\underset{\text{space}}{=}} -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2} m \omega_0^2 x^2$$

To find the eigenstates and eigenenergies, we need to solve the differential equation

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + \frac{1}{2} m\omega_0^2 x^2 \psi(x) = E\psi(x)$$

$$\Rightarrow \frac{d^2}{dx^2}\psi(x) + \frac{2m}{\hbar^2}\left(E - \frac{1}{2} m\omega_0^2 x^2\right)\psi(x) = 0$$

Now, define the oscillator length $x_0 = \sqrt{\hbar/m\omega_0}$

$$\Rightarrow \frac{d^2}{dx^2}\psi(x) + \left(\frac{2mE}{\hbar^2} - \frac{x^2}{x_0^4}\right)\psi(x) = 0$$

The solutions of this equation are

$$\psi_n(x) = \frac{1}{\sqrt{\sqrt{\pi}\, 2^n n!\, x_0}} e^{-x^2/2x_0^2} H_n(x/x_0), \quad n = 0, 1, 2, 3, \ldots$$

where

$$H_n(y) = (-1)^n e^{y^2} \frac{d^n}{dy^n} e^{-y^2}$$

are the Hermite polynomials of order n and

$$E_n = \left(n+\frac{1}{2}\right)\hbar\omega_0$$

are the corresponding <u>discrete</u> eigen-energies.

$$H_0(y) = 1, \quad H_1(y) = 2y, \quad H_2(y) = 4y^2-2, \ldots$$

<u>Algebraic solution using operators:</u>

Introduce the dimensionless operators

$$\hat{p} = \hat{P}/\sqrt{m\hbar\omega_0}, \quad \hat{q} = \hat{X}\sqrt{m\omega_0/\hbar} = \hat{X}/x_0$$

$$= \hat{P}\frac{x_0}{\hbar} \rightsquigarrow \hat{P} = \frac{\hbar}{x_0}\hat{p}, \quad \hat{X} = \hat{q}x_0$$

We can the rewrite the Hamiltonian as

$$\hat{H} = \frac{1}{2m}\hat{P}^2 + \frac{1}{2}m\omega_0^2\hat{X}^2$$

$$= \frac{1}{2m}\frac{\hbar^2}{x_0^2}\hat{p}^2 + \frac{1}{2}m\omega_0^2 x_0^2\hat{q}^2$$

$$= \frac{\hbar^2}{2m}\frac{m\omega_0}{\hbar}\hat{p}^2 + \frac{1}{2}m\omega_0^2\frac{\hbar}{m\omega_0}\hat{q}^2$$

$$= \frac{1}{2}\hbar\omega_0\left(\hat{p}^2+\hat{q}^2\right)$$

Moreover, we can introduce the (non-Hermitian) operators

$$\hat{a} \equiv \frac{1}{\sqrt{2}}(\hat{q}+i\hat{p}) \quad ; \quad \hat{a}^+ = \frac{1}{\sqrt{2}}(\hat{q}-i\hat{p})$$

and note that

$$\hat{a}^+\hat{a} = \frac{1}{2}(\hat{q}-i\hat{p})(\hat{q}+i\hat{p})$$

$$= \frac{1}{2}(\hat{q}^2+\hat{p}^2+i\hat{q}\hat{p}-i\hat{p}\hat{q})$$

$$= \frac{1}{2}(\hat{q}^2+\hat{p}^2+i[\hat{q},\hat{p}])$$

Moreover, we use that

$$[\hat{q},\hat{p}] = \frac{x_0}{\hbar}\frac{1}{x_0}[\hat{x},\hat{p}] = \frac{1}{\hbar}i\hbar = i$$

so that

$$\hat{a}^+\hat{a} = \frac{1}{2}(\hat{q}^2+\hat{p}^2) - \frac{1}{2}$$

leading us to

$$\hat{H} = \frac{1}{2}\hbar\omega_0(\hat{p}^2+\hat{q}^2) = \hbar\omega_0(\underbrace{\hat{a}^+\hat{a}}_{\equiv \hat{N}}+\frac{1}{2})$$

Commutation relations of $\hat{a}$ and $\hat{a}^+$

$$[\hat{a},\hat{a}^+] = \frac{1}{2}[\hat{q}+i\hat{p},\hat{q}-i\hat{p}] = \frac{1}{2}(i\underbrace{[\hat{p},\hat{q}]}_{-i}-i\underbrace{[\hat{q},\hat{p}]}_{i}) = 1$$

Eigenenergies:

$$\hat{H} = \hbar\omega_0 (\hat{a}^\dagger \hat{a} + \tfrac{1}{2}) \quad \text{and} \quad \hat{N} = \hat{a}^\dagger \hat{a}$$

have a joint set of eigenstates since they commute, i.e.

$$\hat{H}|n\rangle = E_n|n\rangle \quad \& \quad \hat{N}|n\rangle = n|n\rangle$$

$$= \hbar\omega_0 (\hat{N} + \tfrac{1}{2})$$

Thus $E_n = \hbar\omega_0(n+\frac{1}{2})$ and we now have to determine the possible values of n.

To this end, we first note that

$$[\hat{a}, \hat{H}] = \hbar\omega_0 [\hat{a}, \hat{a}^\dagger \hat{a}]$$

$$= \hbar\omega_0 \left(\hat{a}\hat{a}^\dagger \hat{a} - \hat{a}^\dagger \hat{a} \hat{a} \right)$$

$$= \hbar\omega_0 \left(\hat{a}\hat{a}^\dagger - \hat{a}^\dagger \hat{a} \right) \hat{a} = \hbar\omega_0 \underbrace{[\hat{a}, \hat{a}^\dagger]}_{1} \hat{a} = \hbar\omega_0 \hat{a}$$

and

$$[\hat{a}^\dagger, \hat{H}] = \hbar\omega_0 [\hat{a}^\dagger, \hat{a}^\dagger \hat{a}]$$

$$= \hbar\omega_0 \left(\hat{a}^\dagger \hat{a}^\dagger \hat{a} - \hat{a}^\dagger \hat{a} \hat{a}^\dagger \right)$$

$$= \hbar\omega_0\, \hat{a}^\dagger [\hat{a}^\dagger, \hat{a}] = -\hbar\omega_0 \hat{a}^\dagger$$

Now, we have

$$\hat{H}\left(\hat{a}|n\rangle\right) = \left(\hat{a}\hat{H} - \hbar\omega_0\hat{a}\right)|n\rangle$$

$$= \left(\hat{a}E_n - \hbar\omega_0\hat{a}\right)|n\rangle$$

$$= \left(E_n - \hbar\omega_0\right)\hat{a}|n\rangle$$

and

$$\hat{H}\left(\hat{a}^+|n\rangle\right) = \left(\hat{a}^+\hat{H} + \hbar\omega_0\hat{a}^+\right)|n\rangle$$

$$= \left(\hat{a}^+E_n + \hbar\omega_0\hat{a}^+\right)|n\rangle$$

$$= \left(E_n + \hbar\omega_0\right)\hat{a}^+|n\rangle$$

We note that $\hat{a}$ and $\hat{a}^+$ create new eigenstates with eigenenergies $E_n \pm \hbar\omega_0$.

In addition, we have

$$[\hat{N}, \hat{a}] = [\hat{a}^+\hat{a}, \hat{a}]$$

$$= \hat{a}^+\hat{a}\hat{a} - \hat{a}\hat{a}^+\hat{a}$$

$$= [\hat{a}^+, \hat{a}]\hat{a} = -\hat{a}$$

and

$$[\hat{N}, \hat{a}^+] = [\hat{a}^+\hat{a}, \hat{a}^+]$$

$$= \hat{a}^+\hat{a}\hat{a}^+ - \hat{a}^+\hat{a}^+\hat{a}$$

$$= \hat{a}^+[\hat{a}, \hat{a}^+] = \hat{a}^+$$

We then see that

$$\hat{N}\hat{a}|n\rangle = (\hat{a}\hat{N} - \hat{a})|n\rangle = (n-1)\hat{a}|n\rangle$$

and

$$\hat{N}\hat{a}^{+}|n\rangle = (\hat{a}^{+}\hat{N} + \hat{a}^{+})|n\rangle = (n+1)\hat{a}^{+}|n\rangle$$

→ $\hat{a}$ and $\hat{a}^{+}$ create new eigenstates of $\hat{N}$ with eigenvalues $n \pm 1$, e.g.

normalization constant.

$$\hat{a}|n\rangle = C_n|n-1\rangle$$

$$\Rightarrow \left(\langle n|\hat{a}^{+}\right)\left(\hat{a}|n\rangle\right) = |C_n|^2\langle n-1|n-1\rangle = |C_n|^2$$

$$\& \quad \langle n|\hat{a}^{+}\hat{a}|n\rangle = n\langle n|n\rangle = n \quad \Rightarrow \quad \underline{|C_n|^2 = n}$$

Since $|C_n|^2 \geq 0$, we also have $n \geq 0$

Moreover; $\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$

By repeated use of $\hat{a}$, we get

$$\hat{a}\hat{a}|n\rangle = \sqrt{n}\,\hat{a}|n-1\rangle = \sqrt{n}\sqrt{n-1}\,|n-2\rangle, \text{ etc.}$$

If n is integer, the sequence ends with

$$\hat{a}|0\rangle = \sqrt{0}\,|n-1\rangle = 0$$

If n is not an integer, we get negative values of n, which is not allowed. ⇒ n must be an integer.

Similar reasoning leads to

$$\hat{a}^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

We then conclude that

$$E_n = (n + \underbrace{\tfrac{1}{2})\hbar\omega_0}_{\text{zero-point energy.}} \quad , \quad n = 0, 1, 2, \ldots$$

→ quantization of energy, cf. Planck's radiation law!

Eigenstates: $|n\rangle = \frac{1}{\sqrt{n!}}(a^{+})^{n}|0\rangle$

Eigenstates in position space, see Zettili, p. 244-246

Coherent states ("most classical states")

Recall that $\hat{a} = \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p})$ and $\hat{a}^{+} = \frac{1}{\sqrt{2}}(\hat{q} - i\hat{p})$, where $\hat{q} = \hat{X}/x_0 = \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^{+})$ and $\hat{p} = \frac{x_0}{\hbar}\hat{P} = \frac{i}{\sqrt{2}}(\hat{a}^{+} - \hat{a})$

with $x_0 = \sqrt{\hbar/m\omega_0}$.

For the groundstate of the harmonic oscillator, we have

$$\langle \hat{X} \rangle = x_0 \langle 0 | \hat{q} | 0 \rangle = \frac{x_0}{\sqrt{2}} \langle 0 | (\hat{a} + \hat{a}^{+}) | 0 \rangle = 0$$

and

$$\langle \hat{P} \rangle = \frac{\hbar}{x_0} \langle 0 | \hat{p} | 0 \rangle = \frac{i\hbar}{\sqrt{2}\, x_0} \langle 0 | (\hat{a}^{+} - \hat{a}) | 0 \rangle = 0$$

Moreover, for the variance we have

$$\langle \hat{X}^2 \rangle = \frac{x_0^2}{2} \langle 0 | (\hat{a} + \hat{a}^{+})^2 | 0 \rangle = \frac{x_0^2}{2} \langle 0 | \hat{a}\hat{a}^{+} | 0 \rangle = \frac{x_0^2}{2}$$

and

$$\langle \hat{P}^2 \rangle = \frac{-\hbar^2}{2x_0^2} \langle 0 | (\hat{a}^{+} - \hat{a})^2 | 0 \rangle = -\frac{\hbar^2}{2x_0^2} \langle 0 | -\hat{a}\hat{a}^{+} | 0 \rangle = \frac{\hbar^2}{2x_0^2}$$

We then obtain

$$\Delta X \Delta P = \sqrt{\frac{x_0^2}{2}} \times \sqrt{\frac{\hbar^2}{2x_0^2}} = \frac{\hbar}{2} \rightsquigarrow \text{minimal uncertainty.}$$

Now, define the unitary operator

$$\hat{D}(z) \equiv e^{z\hat{a}^{\dagger} - z^{*}\hat{a}} \quad \text{for } z \in \mathbb{C}$$

We see that

$$\hat{D}^{\dagger}(z) = e^{z^{*}\hat{a} - z\hat{a}^{\dagger}} = e^{-(z\hat{a}^{\dagger} - z^{*}\hat{a})},$$

showing that we indeed have

$$\hat{D}(z)\,\hat{D}^{\dagger}(z) = e^{z\hat{a}^{\dagger} - z^{*}\hat{a}}\, e^{-(z\hat{a}^{\dagger} - z^{*}\hat{a})} = \mathbb{1}.$$

We can also express the operator as

$$\begin{aligned}
\hat{D}(z) &= e^{z\hat{a}^{\dagger} - z^{*}\hat{a}} \\
&= e^{z\frac{1}{\sqrt{2}}(\hat{q} - i\hat{p}) - z^{*}\frac{1}{\sqrt{2}}(\hat{q} + i\hat{p})} \\
&= e^{\frac{1}{\sqrt{2}}\left[(z - z^{*})\hat{q} - i(z + z^{*})\hat{p}\right]} \\
&= e^{\frac{1}{\sqrt{2}}\left[2i\,\mathrm{Im}\,z\,\hat{q} - 2i\,\mathrm{Re}\,z\,\hat{p}\right]} \\
&= e^{i\sqrt{2}\left[\mathrm{Im}\,z\,\frac{\hat{X}}{x_0} - \mathrm{Re}\,z\,\frac{x_0}{\hbar}\hat{P}\right]} ; \quad x_0 = \sqrt{\frac{\hbar}{m\omega_0}} \\
&= e^{i\left[\mathrm{Im}\,z\sqrt{\frac{2m\omega_0}{\hbar}}\hat{X} - \mathrm{Re}\,z\sqrt{\frac{2}{\hbar m\omega_0}}\hat{P}\right]} \\
&= e^{\frac{i}{\hbar}\left[\mathrm{Im}\,z\sqrt{2\hbar m\omega_0}\,\hat{X} - \mathrm{Re}\,z\sqrt{\frac{2\hbar}{m\omega_0}}\hat{P}\right]} \\
&\equiv e^{\frac{i}{\hbar}\left[p_0\hat{X} - q_0\hat{P}\right]} \equiv \hat{D}(p_0, q_0)
\end{aligned}$$

In the following, we use the notations $\hat{D}(z)$ and $\hat{D}(p_0, q)$ interchangeably.

We now define the coherent states as

$$|z\rangle \equiv \hat{D}(z)\,|n=0\rangle, \quad z \in \mathbb{C}$$

To understand this definition, let us consider the expectation values $\langle \hat{X} \rangle$, $\langle \hat{P} \rangle$, $\langle \hat{X}^2 \rangle$, and $\langle \hat{P}^2 \rangle$.

For example, we have

$$\langle z | \hat{X} | z \rangle = \langle 0 | \hat{D}^{+}(z)\, \hat{X}\, \hat{D}(z) | 0 \rangle$$

To evaluate this expectation value, we have to calculate operator valued functions of the form

$$\hat{f}(x) = e^{x\hat{A}}\, \hat{B}\, e^{-x\hat{A}}$$

To this end, we use a Taylor expansion of $\hat{f}(x)$:

$$\hat{f}(x) = \hat{f}(0) + \hat{f}'(0)\,x + \tfrac{1}{2}\hat{f}''(0)\,x^2 + \ldots,$$

where $\hat{f}(0) = \hat{B}$, $\hat{f}'(0) = \hat{A}\hat{f}(x) - \hat{f}(x)\hat{A}\big|_{x\to 0} = [\hat{A}, \hat{B}]$

$$\hat{f}''(0) = \hat{A}\,[\hat{A}, \hat{B}] - [\hat{A}, \hat{B}]\,\hat{A} = [\hat{A}, [\hat{A}, \hat{B}]],$$

etc, e.g. $\hat{f}'''(0) = [\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]]$

Using this expression, we find

$$\hat{D}^\dagger(z)\,\hat{X}\,\hat{D}(z) = e^{-\frac{i}{\hbar}(p_0\hat{X}-q_0\hat{P})}\,\hat{X}\,e^{\frac{i}{\hbar}(p_0\hat{X}-q_0\hat{P})}$$

$$= \hat{X} - \frac{i}{\hbar}\left[p_0\hat{X}-q_0\hat{P},\hat{X}\right] + \dots$$

$$= \hat{X} + \frac{i}{\hbar}q_0\left[\hat{P},\hat{X}\right] + \dots$$

$$= \hat{X} + \frac{i}{\hbar}q_0(-i\hbar) + 0$$

since $[\hat{P},\hat{X}]$ = c-number

$$= \hat{X} + q_0$$

Thus, the operator $\hat{D}(z)$ displaces the coordinate $\hat{X}$ by the amount q_0, so that

$$\langle z|\hat{X}|z\rangle = \langle 0|\hat{D}^\dagger(z)\,\hat{X}\,\hat{D}(z)|0\rangle$$

$$= \langle 0|\hat{X}+q_0|0\rangle$$

$$= \underbrace{\langle 0|\hat{X}|0\rangle}_{0} + q_0\underbrace{\langle 0|0\rangle}_{1} = q_0$$

For the variance, we similarly find

$$\langle z|\hat{X}^2|z\rangle = \langle 0|\hat{D}^\dagger(z)\,\hat{X}\,\underbrace{\hat{D}(z)\hat{D}^\dagger(z)}_{1}\hat{X}\,|\hat{D}(z)|0\rangle$$

$$= \langle 0|(\hat{X}+q_0)(\hat{X}+q_0)|0\rangle$$

$$= \langle 0|\hat{X}^2|0\rangle + 2q_0\langle 0|\hat{X}|0\rangle + q_0^2\langle 0|0\rangle$$

$$= x_0^2/2 + 0 + q_0^2$$

We then find

$$\Delta X = \sqrt{\langle \hat{x}^2\rangle - \langle \hat{x}\rangle^2}$$

$$= \sqrt{x_0^2/2 + q_0^2 - q_0^2} = x_0/\sqrt{2}$$

For the momentum, we get

$$\langle z|\hat{p}|z\rangle = \langle 0|\hat{D}^\dagger(z)\,\hat{p}\,\hat{D}(z)|0\rangle$$

$$= \langle 0|\left(\hat{p} - \tfrac{i}{\hbar}[p_0\hat{X} - q_0\hat{P}, \hat{p}]\right)|0\rangle$$

$$= \langle 0|\hat{p}|0\rangle + p_0\langle 0|0\rangle = p_0$$

and

$$\langle z|\hat{p}^2|z\rangle = \langle 0|\hat{D}^\dagger(z)\,\hat{p}\,\hat{D}(z)\,\hat{D}^\dagger(z)\,\hat{p}\,\hat{D}(z)|0\rangle$$

$$= \langle 0|(\hat{p}+p_0)(\hat{p}+p_0)|0\rangle$$

$$= \langle 0|\hat{p}^2|0\rangle + 2p_0\langle 0|\hat{p}|0\rangle + p_0^2\langle 0|0\rangle$$

$$= \frac{\hbar^2}{2x_0^2} + p_0^2$$

We then find

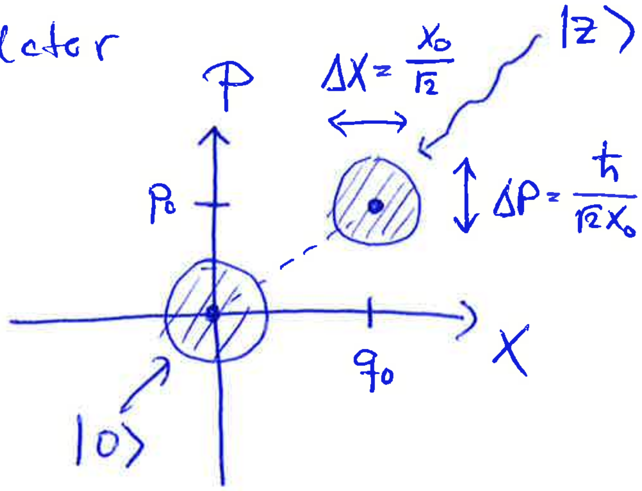
$$\Delta P = \sqrt{\langle \hat{p}^2\rangle - \langle \hat{p}\rangle^2} = \sqrt{\frac{\hbar^2}{2x_0^2} + p_0^2 - p_0^2} = \frac{\hbar}{\sqrt{2}\,x_0}$$

Thus, for the coherent states we again find

$$\Delta X \Delta P = \frac{x_0}{\sqrt{2}}\frac{\hbar}{\sqrt{2}\,x_0} = \frac{\hbar}{2}$$ and minimal uncertainty

We can now think of a coherent state by representing it in the phase space of the oscillator

$$|z\rangle = \hat{D}(p_0, q_0)\,|0\rangle$$



The coherent states are eigenstates of the annihilation operator $\hat{a}$. To see this, we use that

$$\hat{D}(z)\,\hat{a}\,|0\rangle = 0$$

$$= \underbrace{\hat{D}(z)\,\hat{a}\,\hat{D}^\dagger(z)}\,\hat{D}(z)\,|0\rangle$$

$$= e^{z\hat{a}^\dagger - z^*\hat{a}}\,\hat{a}\,e^{-(z\hat{a}^\dagger - z^*\hat{a})}$$

$$= \hat{a} + [z\hat{a}^\dagger - z^*\hat{a}, \hat{a}] + \dots$$

$$= \hat{a} + z\underbrace{[\hat{a}^\dagger, \hat{a}]}_{-1} + 0 = \hat{a} - z$$

We then have

$$0 = (\hat{a} - z)\,\hat{D}(z)\,|0\rangle$$

$$\Rightarrow \hat{a}\,\hat{D}(z)\,|0\rangle = z\,\hat{D}(z)\,|0\rangle$$

We note that the eigenvalue z can be complex, since $\hat{a}$ is not hermitian.

Fock state representation of a coherent state:

We now want to represent a coherent state in the occupation number basis and thus write

$$|z\rangle = \underbrace{\sum_n |n\rangle\langle n|}_{1} z\rangle = \sum_n c_n |n\rangle,$$

where $c_n = \langle n|z\rangle$ are the expansion coefficients. We find the c_n using the eigenvalue equation

$$\hat{a}\,|z\rangle = z\,|z\rangle$$

$$\Rightarrow \quad \langle n|\hat{a}|z\rangle = z\,\langle n|z\rangle$$

$$= \langle n+1|z\rangle \sqrt{n+1} = z\,\langle n|z\rangle$$

$$\Rightarrow \quad \langle n+1|z\rangle = \frac{z}{\sqrt{n+1}}\langle n|z\rangle$$

$$\Rightarrow \quad \langle n|z\rangle = \frac{z}{\sqrt{n}}\langle n-1|z\rangle$$

$$= \frac{z}{\sqrt{n}}\frac{z}{\sqrt{n-1}}\langle n-2|z\rangle$$

$$= \dots = \frac{z^n}{\sqrt{n!}}\langle 0|z\rangle$$

$$\Rightarrow \quad |z\rangle = \langle 0|z\rangle \sum_n \frac{z^n}{\sqrt{n!}}|n\rangle$$

Finally, we need to find $\langle 0|z\rangle = \langle 0|\hat{D}(z)|0\rangle$

To this end, we need that

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-[\hat{A},\hat{B}]/2} \quad \text{if } [\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$$

To show this, we consider the function

$$\hat{g}(x) = e^{x\hat{A}} e^{x\hat{B}}$$

for which

$$\frac{d}{dx}\hat{g}(x) = \hat{A}\,\hat{g}(x) + e^{x\hat{A}}\hat{B}e^{x\hat{B}}$$

$$= \hat{A}\,\hat{g}(x) + e^{x\hat{A}}\hat{B}e^{-x\hat{A}}e^{x\hat{A}}e^{x\hat{B}}$$

$$= \left[\hat{A} + e^{x\hat{A}}\hat{B}e^{-x\hat{A}}\right]\hat{g}(x)$$

$$= \left(\hat{A} + \hat{B} + x[\hat{A},\hat{B}] + 0\right)\hat{g}(x)$$

$$\text{if } [\hat{A},[\hat{A},\hat{B}]] = 0$$

Now, if also $[\hat{B},[\hat{A},\hat{B}]] = 0$, we can solve this differential equation and obtain

$$\hat{g}(x) = e^{(\hat{A}+\hat{B})x + \frac{1}{2}[\hat{A},\hat{B}]x^2} = e^{(\hat{A}+\hat{B})x} e^{\frac{1}{2}[\hat{A},\hat{B}]x^2}$$

check: $\hat{g}'(x) = (\hat{A}+\hat{B})\,\hat{g}(x) + [\hat{A},\hat{B}]x\,\hat{g}(x)$

$$= \left(\hat{A}+\hat{B} + [\hat{A},\hat{B}]x\right)\hat{g}(x) \quad \checkmark$$

Setting $x=1$, we then get

$$\hat{g}(1) = e^{\hat{A}} e^{\hat{B}} = e^{(\hat{A}+\hat{B})} e^{[\hat{A},\hat{B}]/2}$$

We can then write the displacement operator as

$$\mathcal{D}(z) = e^{z\hat{a}^\dagger - z^*\hat{a}} = e^{z\hat{a}^\dagger} e^{-z^*\hat{a}} e^{-\underbrace{[\hat{a}^\dagger, \hat{a}]}_{-1} z(-z^*)/2}$$

$$= e^{-|z|^2/2} e^{z\hat{a}^\dagger} e^{-z^*\hat{a}}$$

We then have

$$\langle 0|z\rangle = e^{-|z|^2/2} \langle 0| e^{z\hat{a}^\dagger} e^{-z^*\hat{a}} |0\rangle$$

$$= e^{-|z|^2/2} \underbrace{\langle 0|1|0\rangle}_{1} = e^{-|z|^2/2}$$

We now arrive at

$$|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle$$

The probability to find the oscillator with m excitations is then

$$P(m) = |\langle m|z\rangle|^2$$

and $\langle m|z\rangle = e^{-|z|^2/2} \frac{z^m}{\sqrt{m!}}$ such that

$$P(m) = \frac{e^{-|z|^2} (|z|^2)^m}{m!}.$$

Now, since $\langle z| \hat{n} |z\rangle = \langle z| \hat{a}^\dagger \hat{a} |z\rangle = z^* z \langle z|z\rangle$
$= |z|^2 \equiv \bar{n}$,

we get the Poisson distribution

$$\boxed{P(m) = \frac{\bar{n}^m}{m!} e^{-\bar{n}}}$$

Time-evolution of coherent states:

$$
\begin{aligned}
|z(t)\rangle &= e^{-i\hat{H}t/\hbar}\,|z\rangle \\
&= e^{-i\hat{H}t/\hbar}\,\hat{D}(z)\,|0\rangle \\
&= e^{-i\omega_0 t/2}\, e^{-i\omega_0 \hat{a}^\dagger \hat{a} t}\, e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\,|n\rangle \\
&= e^{-i\omega_0 t/2}\, e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\, e^{-i\omega_0 \underbrace{\hat{a}^\dagger \hat{a}}_{n} t}\,|n\rangle \\
&= e^{-i\omega_0 t/2}\, e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{(z e^{-i\omega_0 t})^n}{\sqrt{n!}}\,|n\rangle \\
&= e^{-i\omega_0 t/2}\, \hat{D}(z e^{-i\omega_0 t})\,|0\rangle \\
&= e^{-i\omega_0 t/2}\, |z e^{-i\omega_0 t}\rangle
\end{aligned}
$$

Thus, the time-evolution of a coherent state can be described by the time-dependent parameter $z(t) = z e^{-i\omega_0 t} = |z|\, e^{i(\varphi_0 - \omega_0 t)}$

We moreover have

$$
p_0(t) = \sqrt{2\hbar m\omega_0}\,\mathrm{Im}\, z(t) = \sqrt{2\hbar m\omega_0}\,|z|\,\sin(\varphi_0 - \omega_0 t)
$$

$$
q_0(t) = \sqrt{\frac{2\hbar}{m\omega_0}}\,\mathrm{Re}\, z(t) = \sqrt{\frac{2\hbar}{m\omega_0}}\,|z|\,\cos(\varphi_0 - \omega_0 t)
$$