Perturbation theory:

In quantum mechanics, we often encounter problems that can be described by a Hamiltonian on the form

$$\hat{H} = \hat{H}_0 + \lambda \hat{H}_1,$$

where $\hat{H}_0$ describes an easy problem that we can solve, while $\hat{H}_1$ is a perturbation. The perturbation parameter λ must be small, $\lambda \ll 1$, for the perturbation theory to work.

Since we have "solved" $\hat{H}_0$ (it could be, e.g., the harmonic oscillator), we know its eigenvalues $E_n^{(0)}$ and eigenstates $|\psi_n^{(0)}\rangle$, where $\hat{H}_0 |\psi_n^{(0)}\rangle = E_n^{(0)} |\psi_n^{(0)}\rangle$

Below, we assume that these eigenenergies are not degenerate. Moreover, we assume that the eigenvalues and eigenstates can be expanded in the perturbation parameter λ as

$$E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \dots \quad \text{and}$$

$$|\psi_n\rangle = |\psi_n^{(0)}\rangle + \lambda |\psi_n^{(1)}\rangle + \lambda^2 |\psi_n^{(2)}\rangle + \dots$$

We now aim at calculating these corrections order by order in λ. To this end, we write the Schrödinger equation as

$$\hat{H} |\psi_n\rangle = E_n |\psi_n\rangle$$

$$\rightarrow (\hat{H}_0 + \lambda \hat{H}_1)\left(|\psi_n^{(0)}\rangle + \lambda |\psi_n^{(1)}\rangle + \lambda^2 |\psi_n^{(2)}\rangle + \dots \right) =$$

$$\left(E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \dots \right) \times$$

$$\left(|\psi_n^{(0)}\rangle + \lambda |\psi_n^{(1)}\rangle + \lambda^2 |\psi_n^{(2)}\rangle + \dots \right)$$

and collect terms to same order in λ:

$$\lambda^0: \quad \hat{H}_0 |\psi_n^{(0)}\rangle = E_n^{(0)} |\psi_n^{(0)}\rangle$$

$$\lambda^1: \quad \hat{H}_0 |\psi_n^{(1)}\rangle + \hat{H}_1 |\psi_n^{(0)}\rangle = E_n^{(0)} |\psi_n^{(1)}\rangle + E_n^{(1)} |\psi_n^{(0)}\rangle$$

$$\lambda^2: \quad \hat{H}_0 |\psi_n^{(2)}\rangle + \hat{H}_1 |\psi_n^{(1)}\rangle = E_n^{(0)} |\psi_n^{(2)}\rangle + E_n^{(1)} |\psi_n^{(1)}\rangle + E_n^{(2)} |\psi_n^{(0)}\rangle$$

We are free normalize $|\psi_n\rangle$ so that $\langle \psi_n^{(0)} | \psi_n \rangle = 1$

or

$$\underbrace{\langle \psi_n^{(0)} | \psi_n^{(0)} \rangle}_{1} + \lambda \langle \psi_n^{(0)} | \psi_n^{(1)} \rangle + \lambda^2 \langle \psi_n^{(0)} | \psi_n^{(2)} \rangle + \dots = 1$$

for all $\lambda \ll 1$

$$\Rightarrow \langle \psi_n^{(0)} | \psi_n^{(1)} \rangle = \langle \psi_n^{(0)} | \psi_n^{(2)} \rangle = \dots = 0$$

We now find

$$\langle \psi_n^{(0)} | \hat{H}_0 | \psi_n^{(1)} \rangle + \langle \psi_n^{(0)} | \hat{H}_1 | \psi_n^{(0)} \rangle =$$

$$E_n^{(0)} \langle \psi_n^{(0)} | \psi_n^{(1)} \rangle + E_n^{(1)} \langle \psi_n^{(0)} | \psi_n^{(0)} \rangle$$

$$= E_n^{(0)} \underbrace{\langle \psi_n^{(0)} | \psi_n^{(1)} \rangle}_{0} + \langle \psi_n^{(0)} | \hat{H}_1 | \psi_n^{(0)} \rangle = E_n^{(0)} \times 0 + E_n^{(1)} \times 1$$

$$\Rightarrow \boxed{E_n^{(1)} = \langle \psi_n^{(0)} | \hat{H}_1 | \psi_n^{(0)} \rangle}$$

We can also find the first-order correction to the eigenstate. To this end, we rewrite it as

$$|\psi_n^{(1)}\rangle = \underbrace{\left(\sum_m |\psi_m^{(0)}\rangle\langle\psi_m^{(0)}| \right)}_{=1} |\psi_n^{(1)}\rangle$$

$$= \sum_{m \neq n} \langle \psi_m^{(0)} | \psi_n^{(1)} \rangle \, |\psi_m^{(0)}\rangle ;$$

Since $\langle \psi_n^{(0)} | \psi_n^{(1)} \rangle = 1$. We now need to determine the coefficients $\langle \psi_m^{(0)} | \psi_n^{(1)} \rangle$.

For that, we use that

$$\langle \psi_m^{(0)} | \hat{H}_0 | \psi_n^{(1)} \rangle + \langle \psi_m^{(0)} | \hat{H}_1 | \psi_n^{(0)} \rangle =$$

$$E_n^{(0)} \langle \psi_m^{(0)} | \psi_n^{(1)} \rangle + E_n^{(1)} \langle \psi_m^{(0)} | \psi_n^{(0)} \rangle =$$

$$E_m^{(0)} \langle \psi_m^{(0)} | \psi_n^{(1)} \rangle + \langle \psi_m^{(0)} | \hat{H}_1 | \psi_n^{(0)} \rangle = E_n^{(0)} \langle \psi_m^{(0)} | \psi_n^{(1)} \rangle + E_n^{(1)}$$

$$\Rightarrow \quad \langle \psi_m^{(0)} | \psi_n^{(1)} \rangle = \frac{\langle \psi_m^{(0)} | \hat{H}_1 | \psi_n^{(0)} \rangle}{E_n^{(0)} - E_m^{(0)}}$$

so that

$$| \psi_n^{(1)} \rangle = \sum_{m \neq n} \frac{\langle \psi_m^{(0)} | \hat{H}_1 | \psi_n^{(0)} \rangle}{E_n^{(0)} - E_m^{(0)}} | \psi_m^{(0)} \rangle$$

With the correction to the eigenstate, we can find the second-order correction to the eigenenergy. To this end, we use that

$$\langle \psi_n^{(0)} | \hat{H}_0 | \psi_n^{(2)} \rangle + \langle \psi_n^{(0)} | \hat{H}_1 | \psi_n^{(1)} \rangle =$$

$$E_n^{(0)} \underbrace{\langle \psi_n^{(0)} | \psi_n^{(2)} \rangle}_{0} + E_n^{(1)} \underbrace{\langle \psi_n^{(0)} | \psi_n^{(1)} \rangle}_{0} + E_n^{(2)} \underbrace{\langle \psi_n^{(0)} | \psi_n^{(0)} \rangle}_{1}$$

$$= E_n^{(2)} = E_n^{(0)} \underbrace{\langle \psi_n^{(0)} | \psi_n^{(2)} \rangle}_{0} + \langle \psi_n^{(0)} | \hat{H}_1 | \psi_n^{(1)} \rangle$$

$$= \langle \psi_n^{(0)} | \hat{H}_1 \sum_{m \neq n} \frac{\langle \psi_m^{(0)} | \hat{H}_1 | \psi_n^{(0)} \rangle}{E_n^{(0)} - E_m^{(0)}} | \psi_m^{(0)} \rangle$$

$$= \sum_{m \neq n} \frac{\langle \psi_n^{(0)} | H_1 | \psi_m^{(0)} \rangle \langle \psi_m^{(0)} | \hat{H}_1 | \psi_n^{(0)} \rangle}{E_n^{(0)} - E_m^{(0)}}$$

$$\boxed{E_n^{(2)} = \sum_{m \neq n} \frac{| \langle \psi_m^{(0)} | \hat{H}_1 | \psi_n^{(0)} \rangle |^2}{E_n^{(0)} - E_m^{(0)}}}$$

One can in principle continue this perturbation scheme to any order in λ. However, for most applications the first and second order corrections suffice. For that, the matrix elements $\langle \psi_m^{(0)} | \hat{H}_1 | \psi_n^{(0)} \rangle$ should be small, i.e.

$$\left| \frac{\langle \psi_m^{(0)} | H_1 | \psi_n^{(0)} \rangle}{E_n^{(0)} - E_m^{(0)}} \right| \ll 1$$

See textbook for degenerate perturbation theory!

Consider a quantum harmonic oscillator with a quartic perturbation, i.e.

$$\hat{H} = \hat{H}_0 + \hat{H}_1 = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega_0^2 \hat{x}^2 + \lambda \hbar \omega_0 \left(\hat{x}/l_0\right)^4,$$

where $l_0 = \sqrt{\hbar/m\omega_0}$ is the oscillator length,

and $\lambda \ll 1$ is a small perturbation parameter.

The unperturbed Hamiltonian can be written as

$$\hat{H}_0 = \hbar\omega_0 \left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$$

and has the eigenvalues $E_n = \hbar\omega_0 (n+\frac{1}{2})$, $n = 0, 1, 2, \ldots$

and corresponding eigenvectors $|n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^+)^n |0\rangle$.

The perturbation can be written as

$$\hat{H}_1 = \lambda \hbar\omega_0 \frac{1}{l_0^4} \left(\frac{l_0}{\sqrt{2}} (\hat{a}^+ + \hat{a})\right)^4 = \lambda \frac{\hbar\omega_0}{4} (\hat{a}^+ + \hat{a})^4$$

To first order in λ, the corrections to the eigenenergies are

$$E_n^{(1)} = \langle n | \hat{H}_1 | n \rangle = \lambda \frac{\hbar\omega_0}{4} \langle n | (\hat{a}^+ + \hat{a})^4 | n \rangle$$

$$= \lambda \frac{\hbar\omega_0}{4} \langle n | (\hat{a}^+ + \hat{a})^2 \times (\hat{a}^+ + \hat{a})^2 | n \rangle \equiv \lambda \frac{\hbar\omega_0}{4} \langle \tilde{n} | \tilde{n} \rangle$$

with $|\tilde{n}\rangle \equiv (\hat{a}^+ + \hat{a})^2 |n\rangle$

We now find

$$|\tilde{n}\rangle = (\hat{a}^\dagger + \hat{a})^2 |n\rangle$$
$$= \left[\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger + (\hat{a}^\dagger)^2 + \hat{a}^2\right]|n\rangle$$
$$= \left[\hat{a}^\dagger\hat{a} + \hat{a}^\dagger\hat{a} + 1 + (\hat{a}^\dagger)^2 + \hat{a}^2\right]|n\rangle \; ; \quad [\hat{a},\hat{a}^\dagger] = 1$$
$$= (2n+1)|n\rangle + \sqrt{(n+1)(n+2)}\,|n+2\rangle + \sqrt{n(n-1)}\,|n-2\rangle$$

Thus, we get

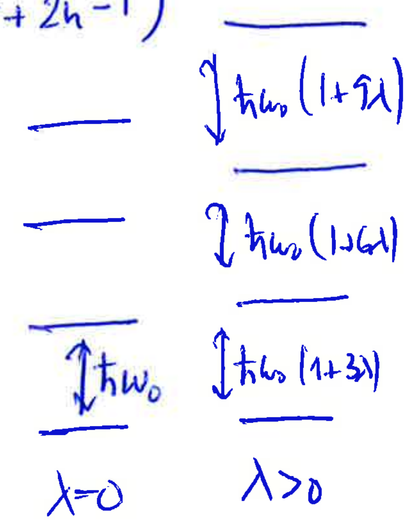
$$\langle\tilde{n}|\tilde{n}\rangle = (2n+1)^2 + (n+1)(n+2) + n(n-1)$$
$$= 4n^2 + 4n + 1 + n^2 + 3n + 2 + n^2 - n$$
$$= 6n^2 + 6n + 3$$

$$\Rightarrow \quad E_n^{(1)} = \lambda \frac{3\hbar\omega_0}{4}(2n^2 + 2n + 1)$$

$$\Rightarrow \quad E_n = \hbar\omega_0\left(n + \tfrac{1}{2}\right) + \lambda\frac{3\hbar\omega_0}{4}(2n^2 + 2n + 1) + \mathcal{O}(\lambda^2)$$

We also find

$$\Delta E_n \equiv E_n - E_{n-1} = \hbar\omega_0\left(1 + \frac{3\lambda}{2}\right) + \hbar\omega_0\frac{3\lambda}{2}\left(n^2 - (n-1)^2\right)$$
$$= \hbar\omega_0\left(1 + \frac{3\lambda}{2}\right) + \hbar\omega_0\frac{3\lambda}{2}\left(n^2 - n^2 + 2n - 1\right)$$
$$= \hbar\omega_0\left(1 + 3\lambda \underline{n}\right)$$



Notice how the energy splitting is no longer constant! Perhaps this can be used for ESR on the groundstate?

The Wentzel-Kramers-Brillouin method (WKB)

The WKB method provides a semi-classical solution to the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi(\underline{r}) + V(\underline{r})\psi(\underline{r}) = E\psi(\underline{r})$$

To start with, we rewrite it as

$$\nabla^2\psi(\underline{r}) + \frac{1}{\hbar^2}2m(E-V(r))\psi(\underline{r}) = 0$$

Recalling the classical expressions $\frac{1}{2}mv^2$ and pv for energy and momentum, we can identify

$$E-V = \frac{1}{2}mv^2 = p^2/2m \Rightarrow$$

$$p = \pm\sqrt{2m(E-V)}$$ with the "classical" momentum if $E>V$

$$\Rightarrow \quad \nabla^2\psi(\underline{r}) + \frac{1}{\hbar^2}p^2(r)\psi(\underline{r}) = 0$$

If the potential $V(\underline{r})$ is constant, the solutions to the Schrödinger equation would be of the form

$$\psi(\underline{r}) = Ae^{\pm i\underline{p}\cdot\underline{r}/\hbar}$$

Now, if $V(\underline{r})$ is slowly varying, the WKB method assumes that the solutions are of the form $\psi(\underline{r}) = A(\underline{r})\,e^{iS(\underline{r})/\hbar}$

Inserting this ansatz into the equation

$$\nabla^2\psi + \frac{1}{\hbar^2}p^2\psi = 0,$$

we find

$$
\begin{aligned}
0 &= \nabla^2\left(Ae^{iS/\hbar}\right) + \frac{1}{\hbar^2}p^2 A e^{iS/\hbar} \\
&= \nabla\cdot\left((\nabla A)e^{iS/\hbar} + Ae^{iS/\hbar}\frac{i}{\hbar}\nabla S\right) + \frac{1}{\hbar^2}p^2 A e^{iS/\hbar} \\
&= (\nabla^2 A)e^{iS/\hbar} + \frac{i}{\hbar}(\nabla A)\cdot(\nabla S)e^{iS/\hbar} \\
&\quad + \frac{i}{\hbar}(\nabla A)\cdot(\nabla S)e^{iS/\hbar} + Ae^{iS/\hbar}\left(\frac{i}{\hbar}\right)^2(\nabla S)^2 \\
&\quad + Ae^{iS/\hbar}\frac{i}{\hbar}\nabla^2 S + \frac{1}{\hbar^2}p^2 A e^{iS/\hbar}
\end{aligned}
$$

$\times \hbar^2 e^{-iS/\hbar}$
$\Rightarrow$

$$
\begin{aligned}
&\hbar^2\nabla^2 A + i\hbar\, 2(\nabla A)\cdot(\nabla S) - A(\nabla S)^2 \\
&+ i\hbar A\nabla^2 S + p^2 A = \\
&A\left(\frac{\hbar^2}{A}\nabla^2 A - (\nabla S)^2 + p^2\right) + i\hbar\left(2(\nabla A)\cdot(\nabla S) + A\nabla^2 S\right) = 0
\end{aligned}
$$

$$\Rightarrow \begin{cases} \frac{\hbar^2}{A}\nabla^2 A - (\nabla S)^2 + p^2 = 0 \\ 2(\nabla A)\cdot(\nabla S) + A\nabla^2 S = 0 \end{cases}$$

$$\Rightarrow \begin{cases} (\nabla S)^2 = p^2 + \frac{\hbar^2}{A}\nabla^2 A \overset{\text{class limit}}{\simeq} p^2 = 2m(E-V) \\ 2(\nabla A)\cdot(\nabla S) + A\nabla^2 S = 0 \end{cases},$$

where we have used $\hbar \sim 0$ in the semi-classical limit.

Here, it has been assumed that $E > V$ such that the momentum p is real.

We now consider a one-dimensional problem, i.e.

$$(\nabla S)^2 \to \left(\frac{dS}{dx}\right)^2 = 2m(E-V) \Rightarrow \frac{dS}{dx} = \pm p = \pm\sqrt{2m(E-V)}$$

and

$$2\left(\frac{d}{dx}A\right)\frac{dS}{dx} + A\frac{d^2}{dx^2}S = 0$$

$$= 2\left(\frac{d}{dx}A\right)(\pm p) + A\frac{d}{dx}(\pm p)$$

$$\Rightarrow \quad 2\frac{1}{A}\frac{d}{dx}A + \frac{1}{p}\frac{d}{dx}p = 0$$

$$\Rightarrow \quad 2\frac{d}{dx}\ln A + \frac{d}{dx}\ln p = \frac{d}{dx}\ln\left(A^2 p\right) = 0$$

$$\Rightarrow \quad A^2 p = \text{const.} \Rightarrow A(x) = \frac{C}{\sqrt{|p(x)|}}$$

Thus, we find solutions of the form

$$\psi_\pm(x) = \frac{C_\pm}{\sqrt{|p(x)|}}\, e^{\pm i\int dx\, p(x)/\hbar},$$

which holds for the classically allowed region, $E > V(x)$.

For the classically forbidden region, $E < V(x)$, a similar analysis is needed, and one finds solutions of the form

$$\psi_\pm(x) = \frac{C_\pm}{\sqrt{|p(x)|}}\, e^{\pm\int dx\, |p(x)|/\hbar}$$

Alternative derivation of the WKB approximation:

$$\frac{d^2}{dx^2}\psi(x) + \frac{1}{\hbar^2}p^2\psi(x) = 0, \quad p^2 = 2m(E-V)$$

We write the solution on the form

$$\psi(x) = e^{if(x)/\hbar},$$

where $f(x)$ is a complex function and we think of $\hbar$ as a small parameter ($\hbar \to 0$).

Inserting this expression above, we find

$$0 = \frac{d}{dx}\left[e^{if/\hbar}\frac{i}{\hbar}f'\right] + \frac{1}{\hbar^2}p^2 e^{if/\hbar}$$

$$= e^{if/\hbar}\left(\frac{i}{\hbar}f'\right)^2 + e^{if/\hbar}\frac{i}{\hbar}f'' + \frac{1}{\hbar^2}p^2 e^{if/\hbar}$$

$$\Rightarrow 0 = -(f')^2 + i\hbar f'' + p^2 = i\hbar f'' - (f')^2 + p^2$$

Now, we expand $f(x)$ in powers of $\hbar$:

$$f(x) = f_0(x) + \hbar f_1(x) + \hbar^2 f_2(x) + \dots$$

$$\Rightarrow 0 = i\hbar\left[f_0'' + \hbar f_1'' + \hbar^2 f_2'' + \dots\right] - \left[f_0' + \hbar f_1' + \hbar^2 f_2' + \dots\right]^2 + p^2$$

$$= i\hbar[\dots] - \left((f_0')^2 + 2\hbar f_0' f_1' + 2\hbar^2 f_0' f_2' + \hbar^2 (f_1')^2 + \dots\right) + p^2$$

Collecting terms to same order in $\hbar$, we find

$\hbar^0$: $(f_0')^2 = p^2$; $\hbar^1$: $if_0'' = 2f_0'f_1'$; $\hbar^2$: $if_1'' = 2f_0'f_2' + (f_1')^2$, etc.

Next, we solve these equations to first order in $\hbar$:

$\hbar^0$: $f_0' = \pm p \Rightarrow f_0 = \pm \int p\,dx$

$\hbar^1$: $i f_0'' = 2 f_0' f_1' \Rightarrow f_1' = \frac{i}{2} \frac{f_0''}{f_0'} = \frac{i}{2} \frac{p'}{p} = \frac{i}{2} \frac{d}{dx} \ln p$

$$\Rightarrow f_1 = \frac{i}{2} \ln p$$

$$= \frac{i}{2} \left[\ln |p| + i \arg p \right]$$

$$= i \ln \sqrt{|p|} - \frac{1}{2} \begin{cases} \pi/2 & \text{if imag} \\ 0 & \end{cases}$$

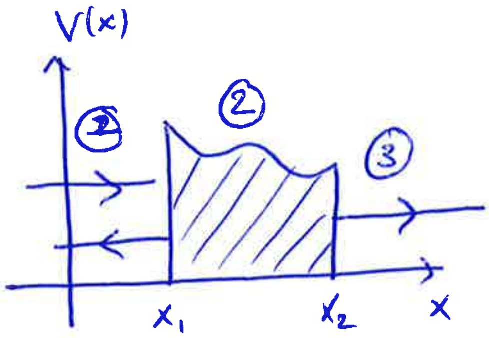
We then have

$$\psi(x) = e^{i f(x)/\hbar} = e^{\frac{i}{\hbar}\left[\pm \int dx\, p + \hbar \left(i \ln \sqrt{|p|} - \frac{\arg p}{2} \right) \right]}$$

$$= \frac{C}{\sqrt{|p|}} e^{\pm i \int p\,dx / \hbar}$$

Example of the WKB approximation: tunneling

Consider an incident particle with momentum $p_0 = \sqrt{2mE}$ hitting the potential barrier sketched in the figure, where $E < V_{max}$



In region ①, the wave function reads

$$\psi_1(x) = \psi_{in}(x) + \psi_{re}(x) = A e^{ip_0x/\hbar} + B e^{-ip_0x/\hbar}$$

In region ③, we have

$$\psi_3(x) = M e^{ip_0x/\hbar}$$

In region ②, we use the WKB approximation and write

$$\psi_2(x) = \frac{C}{\sqrt{|p(x)|}} e^{-\frac{1}{\hbar}\int_{x_1}^{x} dx'|p(x')|} + \underbrace{\frac{D}{\sqrt{|p(x)|}} e^{\frac{1}{\hbar}\int_{x_1}^{x} dx'|p(x')|}}$$

→ blows up with barrier thickness, which is unphysical

→ $D \simeq 0$

$$= \frac{C}{\sqrt{|p(x)|}} e^{-\frac{1}{\hbar}\int_{x_1}^{x} dx'|p(x')|}$$

We now want to evaluate the transmission coeffecient

$$T = \frac{|M|^2}{|A|^2}$$

To this end, we use that the wave function and its first derivative must be continuous at x_1 and x_2

At $x = x_1$, we ~~then~~ have

$$A e^{i p_0 x_1/\hbar} + B e^{-i p_0 x_1/\hbar} = \frac{C}{\sqrt{|p(x_1)|}}$$

and

$$\frac{i}{\hbar} p_0 \left(A e^{i p_0 x_1/\hbar} - B e^{-i p_0 x_1/\hbar} \right) =$$

$$\frac{d}{dx}\left[\frac{C}{\sqrt{|p(x)|}} e^{-\frac{1}{\hbar}\int_{x_1}^{x} dx' \, |p(x')|} \right]\Bigg|_{x \to x_1}$$

$$\simeq \frac{C}{\sqrt{|p(x_1)|}} e^{\underbrace{-\frac{1}{\hbar}\int_{x_1}^{x_1} dx' \, |p(x')|}_{0}} \times -\frac{1}{\hbar} |p(x_1)|$$

$$= -\frac{C}{\hbar} \sqrt{|p(x_1)|},$$

where it has been assumed that $\frac{d}{dx} |p(x)| \big|_{x \to x_1} \simeq 0$

At $x = x_2$, we similarly find

$$\frac{C}{\sqrt{|p(x_2)|}} e^{-\frac{1}{\hbar}\int_{x_1}^{x_2} dx' \, |p(x')|} = M e^{i p_0 x_2/\hbar}$$

$$-\frac{C}{\hbar} \sqrt{|p(x_2)|} \, e^{-\frac{1}{\hbar}\int_{x_1}^{x_2} dx' \, |p(x')|} = \frac{i p_0}{\hbar} M e^{i p_0 x_2/\hbar}$$

From the case $x = x_1$, we get

$$2Ae^{ip_0x_1/\hbar} = \frac{C}{\sqrt{|p(x_1)|}} - \frac{C}{ip_0}\sqrt{|p(x_1)|}$$

$$= \frac{C}{\sqrt{|p(x_1)|}}\left(1 - \frac{|p(x_1)|}{ip_0}\right)$$

$$\Rightarrow \quad C = 2A\sqrt{|p(x_1)|}\, e^{ip_0x_1/\hbar} \Big/ \left(1 - \frac{|p(x_1)|}{ip_0}\right)$$

Inserting this expression into the result for $x = x_2$, we obtain

$$\mathcal{M} e^{ip_0x_2/\hbar} = \frac{C}{\sqrt{|p(x_2)|}}\, e^{-\frac{1}{\hbar}\int_{x_1}^{x_2} dx'\, |p(x')|}$$

$$= 2A\sqrt{\frac{|p(x_1)|}{|p(x_2)|}}\, e^{ip_0x_1/\hbar}\, e^{-\frac{1}{\hbar}\int_{x_1}^{x_2} dx'\, |p(x')|} \Big/ \left(1 - \frac{|p(x_1)|}{ip_0}\right)$$

$$\Rightarrow \quad \frac{\mathcal{M}}{A} = \frac{2}{1 + i\frac{|p(x_1)|}{p_0}}\sqrt{\frac{|p(x_1)|}{|p(x_2)|}}\, e^{ip_0(x_1 - x_2)/\hbar}\, e^{-\frac{1}{\hbar}\int_{x_1}^{x_2} dx'\, |p(x')|}$$

We then arrive at

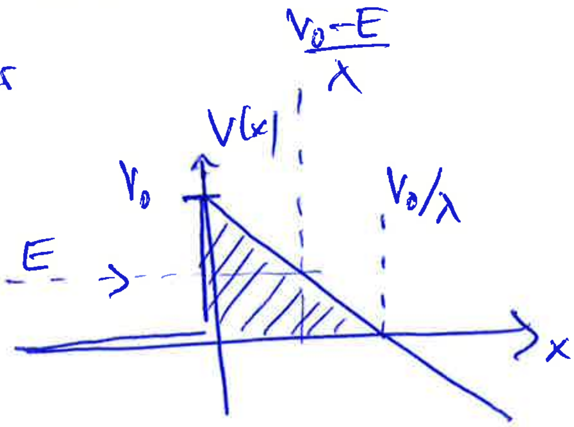
$$T = \frac{|\mathcal{M}|^2}{|A|^2} = \frac{4}{1 + \frac{|p(x_1)|^2}{p_0^2}}\, \frac{|p(x_1)|}{|p(x_2)|}\, e^{-\frac{2}{\hbar}\int_{x_1}^{x_2} dx'\, |p(x')|^2}$$

$$= \frac{4}{|p(x_2)|/|p(x_1)| + |p(x_1)||p(x_2)|/p_0^2}\, e^{-\frac{2}{\hbar}\int_{x_1}^{x_2} dx'\, |p(x')|^2}$$

$$\sim e^{-2\gamma} \quad \text{with} \quad \gamma = \frac{1}{\hbar}\int_{x_1}^{x_2} dx'\, \sqrt{2m(V(x') - E)}$$

As an example, let us consider tunneling through a barrier of the form

$$V(x) = \begin{cases} 0, & x<0 \\ V_0 - \lambda x, & x>0 \end{cases}$$



In a crude approximation, we calculate the tunneling rate γ as

$$\gamma = \frac{1}{\hbar}\int_0^{\frac{V_0-E}{\lambda}} dx \sqrt{2m(V_0-\lambda x-E)}$$

$$= \frac{\sqrt{2m}}{\hbar}\left[\frac{2}{-3\lambda}(V_0-\lambda x-E)^{3/2}\right]_0^{\frac{V_0-E}{\lambda}} \quad \nearrow 0 \quad \to \frac{2}{3\lambda}(V_0-E)^{3/2}$$

$$= \frac{2\sqrt{2m}}{3\hbar\lambda}(V_0-E)^{3/2}$$

$$\to \underline{T \simeq e^{-\frac{4\sqrt{2m}}{3\hbar\lambda}(V_0-E)^{3/2}}}$$