No-cloning theorem:

In many cases it might be useful to copy or clone quantum information. However, due to the linear structure of quantum mechanics, this turns out to be impossible. To see this, imagine that we actually have a unitary operation that allows us to copy a state:

$$\hat{U} \, |\phi\rangle \otimes |\psi\rangle = |\phi\rangle \otimes |\phi\rangle \otimes |\psi'\rangle$$

Now, if we copy two different states, we find

$$\hat{U} \, |\phi_1\rangle \otimes |\psi\rangle = |\phi_1\rangle \otimes |\phi_1\rangle \otimes |\psi'\rangle$$

$$\hat{U} \, |\phi_2\rangle \otimes |\psi\rangle = |\phi_2\rangle \otimes |\phi_2\rangle \otimes |\psi''\rangle$$

However, if we try, to copy the superposition $|\phi_s\rangle = \frac{1}{\sqrt{2}}\left(|\phi_1\rangle + |\phi_2\rangle\right)$, we get

$$\hat{U} \, |\phi_s\rangle \otimes |\psi\rangle = \frac{1}{\sqrt{2}} |\phi_1\rangle \otimes |\phi_1\rangle \otimes |\psi'\rangle + \frac{1}{\sqrt{2}} |\phi_2\rangle \otimes |\phi_2\rangle \otimes |\psi''\rangle$$

which is different from $|\phi_s\rangle \otimes |\phi_s\rangle \otimes |\psi''\rangle$

This contradiction implies the no-cloning theorem.

Quantum teleportation

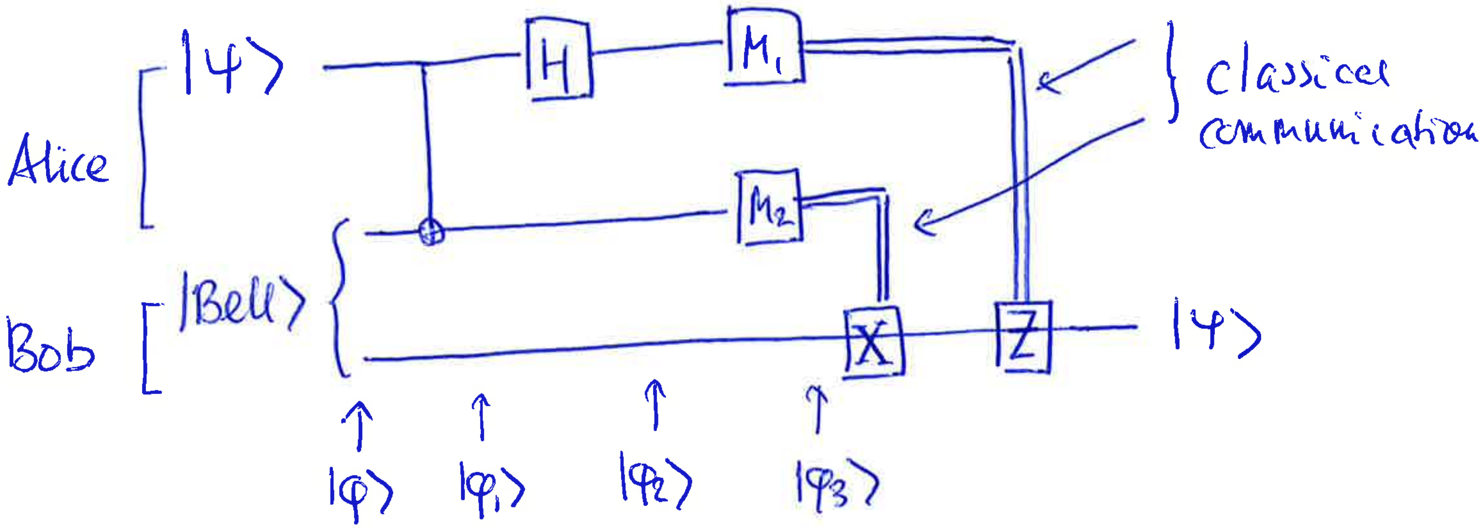
Imagine that Alice has received an unknown qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ that she wants to transfer to Bob who is far away. She can't copy the state because of the no-cloning theorem, and she can't measure it, because it would destroy it.

It turns out that Alice can teleport the qubit to Bob if they share an entangled state! Assume that the entangled state has the form $|Bell\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, where Alice and Bob each have one of the qubits. Now, the combined state is

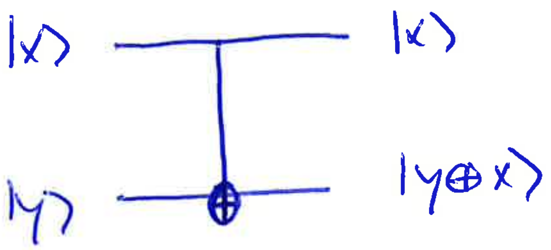
$$|\varphi\rangle = |\psi\rangle|Bell\rangle$$
$$= \frac{1}{\sqrt{2}}\left[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|00\rangle + |11\rangle)\right]$$

Here, the first two qubits belong to Alice, while the third belongs to Bob.

To carry out the quantum teleportation, the following circuit is implemented



First, Alice applies a controlled-NOT (CNOT) gate to her qubits. The CNOT gate flips the second qubit conditioned on the first qubit:



$$\rightarrow \quad \begin{array}{l} |00\rangle \rightarrow |00\rangle \\ |01\rangle \rightarrow |01\rangle \\ |10\rangle \rightarrow |11\rangle \\ |11\rangle \rightarrow |10\rangle \end{array}$$

The CNOT gate is essential in quantum information processing as it generates entanglement, e.g.

$$(\hat{H}_1 \otimes 1)|00\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

After the CNOT gate, we have

$$|\varphi_1\rangle = \frac{1}{\sqrt{2}}\Big(\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|10\rangle + |01\rangle)\Big)$$

Second, a Hadamard gate is applied to the first qubit, and we get

$$|\varphi_2\rangle = \frac{1}{\sqrt{2}}\Big(\alpha \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)(|00\rangle+|11\rangle) + \beta \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)(|10\rangle+|01\rangle)\Big)$$

$$= \frac{1}{2}\Big[\alpha|000\rangle+\alpha|011\rangle+\alpha|100\rangle+\alpha|111\rangle + \beta|010\rangle+\beta|001\rangle-\beta|110\rangle-\beta|101\rangle\Big]$$

$$= \frac{1}{2}\Big[|00\rangle(\alpha|0\rangle+\beta|1\rangle)+|01\rangle(\alpha|1\rangle+\beta|0\rangle) + |10\rangle(\alpha|0\rangle-\beta|1\rangle)+|11\rangle(\alpha|1\rangle-\beta|0\rangle)\Big]$$

In the third step, Alice measures her qubits and communicates her results to Bob. There are 4 possible outcomes:

Alice measures:		Bob does:
$\vert 00\rangle$	$\longrightarrow$	Nothing; his state is already $\alpha\vert 0\rangle+\beta\vert 1\rangle$
$\vert 01\rangle$	$\longrightarrow$	Applies $\hat{\sigma}_x$ to get $\alpha\vert 0\rangle+\beta\vert 1\rangle$
$\vert 10\rangle$	$\longrightarrow$	Applies $\hat{\sigma}_z$ to get $\alpha\vert 0\rangle+\beta\vert 1\rangle$
$\vert 11\rangle$	$\longrightarrow$	Applies $\hat{\sigma}_z\hat{\sigma}_x$ to get $\alpha\vert 0\rangle+\beta\vert 1\rangle$

Thus, in the end, the state $\alpha|0\rangle + \beta|1\rangle$ has been teleported from Alice to Bob. We have not violated the no-cloning theorem, since the original state with Alice was destroyed by the measurement. We have also not violated causality (or faster-than-light transmission), since the measurement outcomes have to be communicated between Alice and Bob via classical means. Finally, this example nicely illustrates how entanglement is an important and necessery ressource for quantum information processing and communication.