Model solutions to old problem set questions

1. At any given price p, **market supply** is the aggregate amount that suppliers can push to the market at cost equal or less than p. Analogously, **market demand** at price p consists of all the units of, say, consumption that generate value higher than p.

   Total expenditure equals equilibrium price times quantity. By **consumer surplus** we mean the difference between your willingness to pay (reservation price) and the actual price you pay. **Producer surplus** is the difference between selling price and cost of producing (reservation price).To calculate the surpluses we'll employ **inverse demand and supply curves**. These essentially depict the same information as their direct counterparts but from a different perspective which allows for easy calculation of surpluses.

   You can think about inverse demand curve $p^D(q)$ like this: order the consumed units from the one that gives the most satisfaction to the one that leaves the buyer indifferent between buying or not. Consumers pay the same price for each consumed unit and therefore consumer surplus is the area between inverse demand curve and equilibrium price. This area is highlighted in blue in Figure 1.

   For inverse supply $p^S(q)$, substitute satisfaction for cost and order from the lowest to the indifferent one. Producer surplus is highlighted in red in Figure 1.

   In the special case of linear supply and demand curves we can calculate total expenditure and surpluses using geometry. In other cases we have to use calculus. Both kind of solutions are provided below.

   (a) Calculate **equilibrium price**:
   $Q^D(p) = Q^S(p) \iff 100 - 2p = 3p - 40 \implies p^* = 28$
   Substitute p^* into either <u>demand</u> or supply curve for **equilibrium quantity**:
   $Q^D(p^*) = 100 - 2 \times 28 \implies Q^* = 44$
   Total expenditure $Q^* \times p^* = 44 \times 28 = 1232$
   Consumer surplus:
   Calculate inverse demand curve: $Q^D(p) = 100 - 2p \iff p^D(q) = 50 - \frac{q}{2}$
   Calculate the area between equilibrium price and inverse demand (=consumer surplus):
   Geometry: $((50 - 28) \times (44 - 0))/2 = 484$
   Integration: $\int_0^{q^*} p^D(q) - p^* dq = 22q - \frac{q^2}{4} = 484$
   Producer surplus:
   Calculate inverse supply curve: $Q^S(p) = 3p - 40 \iff p^D(q) = \frac{40+q}{3}$
   Calculate the area between equilibrium price and inverse supply (=producer surplus):
   Geometry: $((28 - 40/3) \times (44 - 0))/2 = 322\frac{2}{3}$
   Integration: $\int_0^{q^*} p^* - p^S(q) dq = \frac{44+q}{3} - \frac{q^2}{6} = 322\frac{2}{3}$

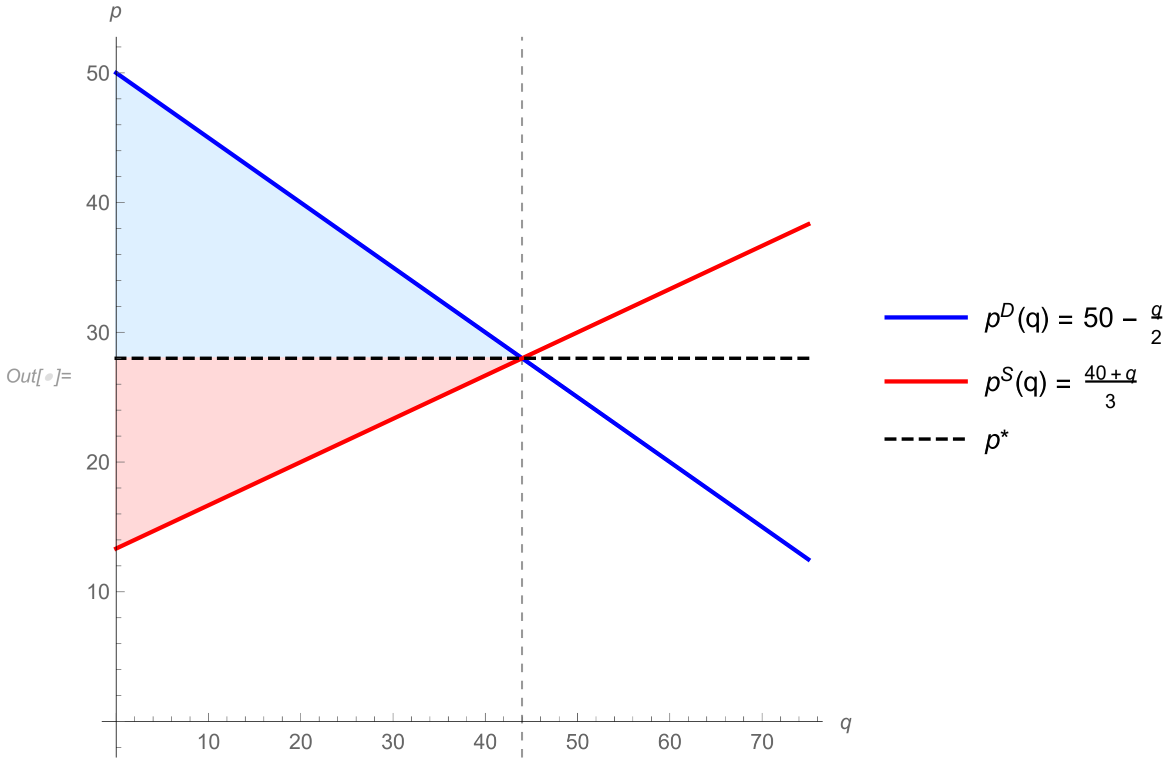


Figure 1: Inverse demand and supply in (a).

(b) Here we have to aggregate supply from 1000 identical producers. An individual firm's supply curve simply states, how much output it will feed to the market at a given price. Market supply thus is obtained by just aggregating these individual supplies together: $Q^S(p) = \sum_{i=1}^{1000} q_i^S(p) = 1000 * (p/200 - q/25) = 5p - 40$.

The requested numbers are obtained with exactly the same steps as in (a):

Equilibrium price:

$Q^D(p) = Q^S(p) \iff 100 - 2p = 5p - 40 \implies p^* = 20$

Equilibrium quantity:

$Q^D(p^*) = 100 - 2 * 20 \implies Q^* = 60$

Total expenditure $Q^* \times p^* = 60 \times 20 = 1200$

Consumer surplus:

Inverse demand curve is unchanged. Geometry: $((50 - 20) \times (60 - 0))/2 = 900$

Integration: $\int_0^{q^*} p^D(q) - p^* dq = \int_0^{q^*} 50 - \frac{q}{2} - 20\ dq = 900$

Producer surplus:

Calculate inverse supply curve: $Q^S(p) = 5p - 40 \iff p^D(q) = 8 + \frac{q}{5}$

Geometry: $((20 - 8) \times (60 - 0))/2 = 360$

Integration: $\int_0^{q^*} p^* - p^S(q)dq = \int_0^{q^*} 20 - 8 - \frac{q}{5}\ dq = 360$

(c) All the 100 suppliers will supply at $p = 28$ and therefore $Q^S(28) = 100$. There will be no suppliers at $p = 8$ and therefore $Q^S(8) = 0$. Suppliers' reservation prices are uniformly distributed between these extremes which implies that the number of suppliers and therefore aggregate supply is linear in price. In other words: at any price $8 < p < 28$ we're equally likely to lose (gain) a producer if we decrease (increase) the price a little.

We're dealing with a linear function and we know two points on that line. It's elementary to find the line equation $Q^S(p) = 5p - 40$. *It turns out to be same as in (b) and therefore all the calculations will be identical.*

More technically, denote the reservation price by P. $P \sim U(8, 28)$. Uniform distribution has constant probability density function, PDF, which in this case stands as $f(p) = 1/20$ if $p \in [8, 28]$, 0 otherwise. PDF being constant means that any value is equally likely. Cumulative density function, CDF, for the reservation price is given by $F(p) = \int_{-\infty}^{p} f(p)\, dp = 0$ if $p < 8$, $\frac{5p-40}{100}$ if $p \in [8, 28]$, 1 otherwise.
In other words, CDF gives the probability that a supplier's reservation price is less than equal to p, which is the probability that the firm produces. Knowing the number of firms (100), the probability that an individual firm is producing at a given price $(F(p))$, and the capacity of single supplier (1) we have the market supply $(Q^S(p) = 5p - 40)$.

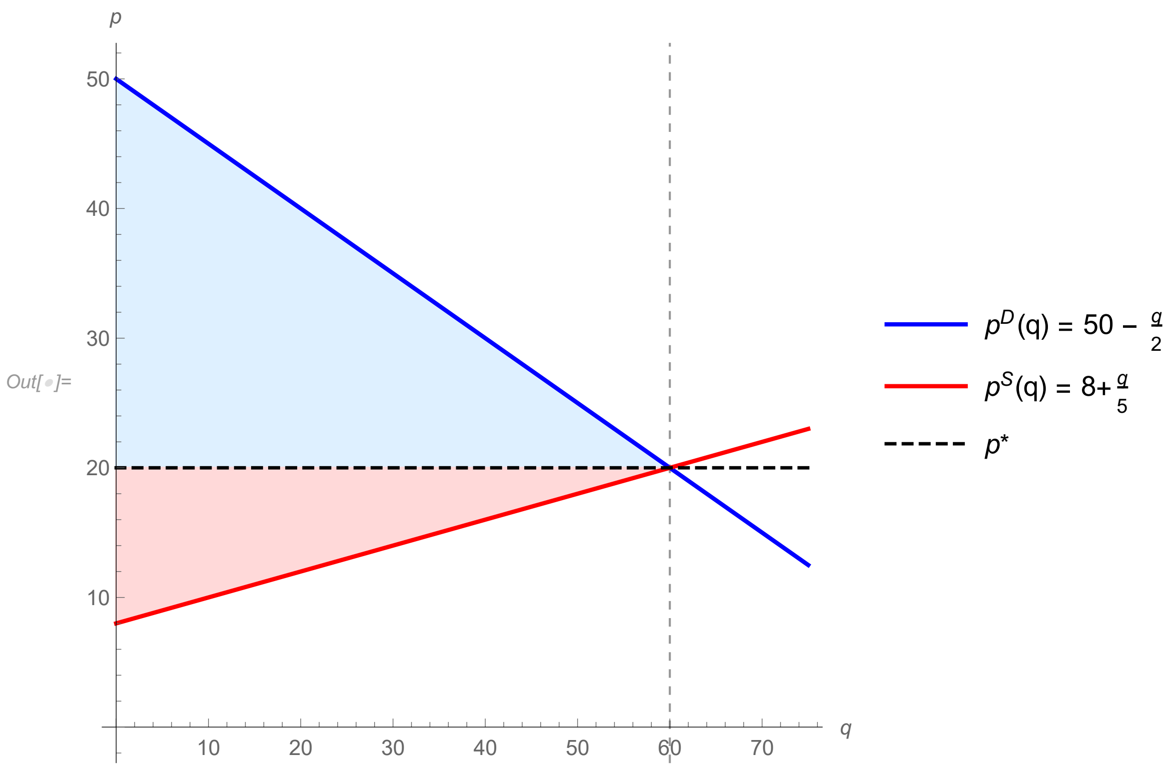


Figure 2: Inverse demand and supply in (b) and (c).

2. The key here is to find the *effective supply and demand curves* in each in the scenarios. All the prices are in euros pre square meter, quantities in 1000's of square meters.

   (a) First we determine the equilibrium price at the current, *short-run fixed supply* of housing and *pre-pandemic demand*. Recall that in the equilibrium, price is such that the demand side is willing to buy the same quantity the supply side is willing to provide: $Q_0^D(p) = Q_0^S(p) \iff 12000 - 3500p = 5000 \implies p^* = 2$. As new houses are only built if $p > 2$, the current quantity is also the long-run equilibrium, $q^* = 5000$. Should the equilibrium price have been any higher, housing stock would've adjusted upwards.

(b) In the short run, prices adjust but *supply is assumed fixed and therefore unchanged from (a)*, $q^* = 5000$. *Demand curve shifts upwards* in the suburb ($Q_0^D(p) \to Q_1^D(p)$) and we have $Q_1^D(p) = Q_0^S(p) \iff 12000 - 2000p = 5000 \implies p^* = 3\frac{1}{2}$ in the post-pandemic short-run equilibrium.

(c) With the *post-pandemic increased demand*, short-run equilibrium price exceeds the threshold above which supply of housing will adjust in the long run, $p^* = 3\frac{1}{2} > 2$. With $p > 2$, $Q_1^S(p) = 5000 + 1000(p-2)$, as the latter part of that equation is just for additional supply exceeding the fixed stock. It's pretty much our standard supply curve but as supply can only adjust upwards in reaction to price changes, we have a kink in the *long run supply curve* at $p = 2$, as seen in Figure 3 and the familiar equivalence of supply and demand is written as $Q_1^D(p) = Q_1^S(p) \iff 12000 - 2000p = 5000 + 1000(p-2) \implies p^* = 3$. Substitute this into demand curve to get $Q_1^D(p^*) = 12000 - 2000 * 3 \implies q^* = 6000$.

(d) The pandemic induced boom in construction in the suburb *increased the inelastic short run supply* of housing from 5000 to 6000 $((Q_0^S(p) \to Q_2^D(p)))$. We're back with our *original demand curve* $(Q_1^D(p) \to Q_0^D(p)))$ and the supply meets demand at $Q_0^D(p) = Q_2^S(p) \implies 12000 - 3500p = 6000 \implies p^* = \frac{12}{7}$.

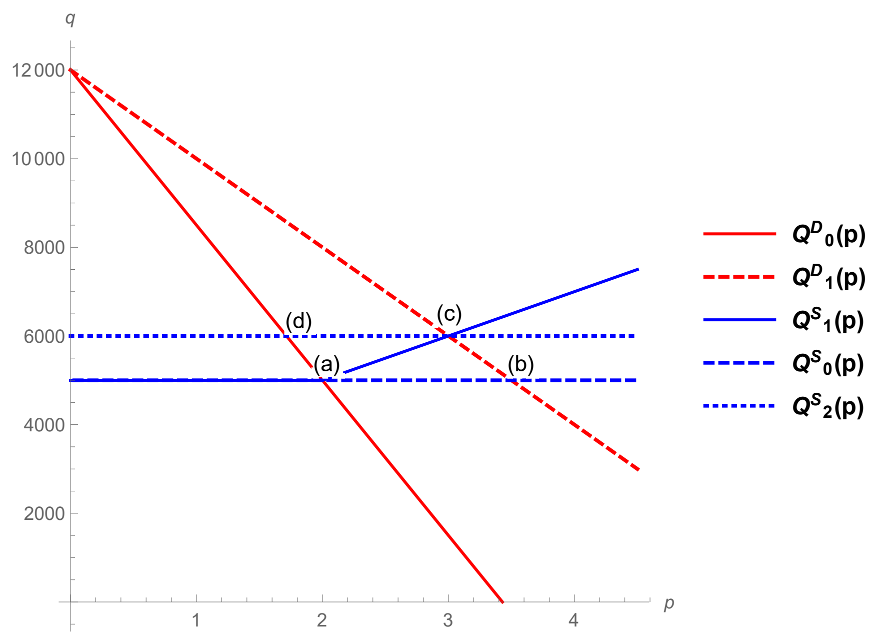


Figure 3: Supply and demand curves in the various scenarios of the housing market.

3. All the valuations are in thousands of euros.

(a) The parties will rationally trade until trades with mutual gains (=Pareto improvements) are not available. This means that eventually the items must be held by those seven who value the items most. Only then no holder will find a trading partner who'd be willing to pay more than the holder's valuation.

Denote the total valuation of item holders in the initial allocation by $U_0 = 80 + 95 + 100 + 120 + 135 + 145 + 200 + 1000 = 1875$. We argued that the allocation after trading must satisfy $U_1 = 1000 + 200 + 180 + 145 + 135 + 120 + 105 + 100 = 1985$. Therefore the surplus created is $U_1 - U_0 = 110$. We need not to know who trades with whom and at which price i.e. individuals' surpluses to obtain total surplus.

(b) Given that our market is able to allocate the items efficiently, i.e. those who value the goods the most actually obtain them, we cannot do any better in terms of total surplus, least by introducing imperfect information or search frictions.

If waiting or searching for a trading partner isn't costly, patient agents will reach the efficient allocation eventually regardless of the pattern of arrivals. In fact, such market essentially coincides with the market in 3a.

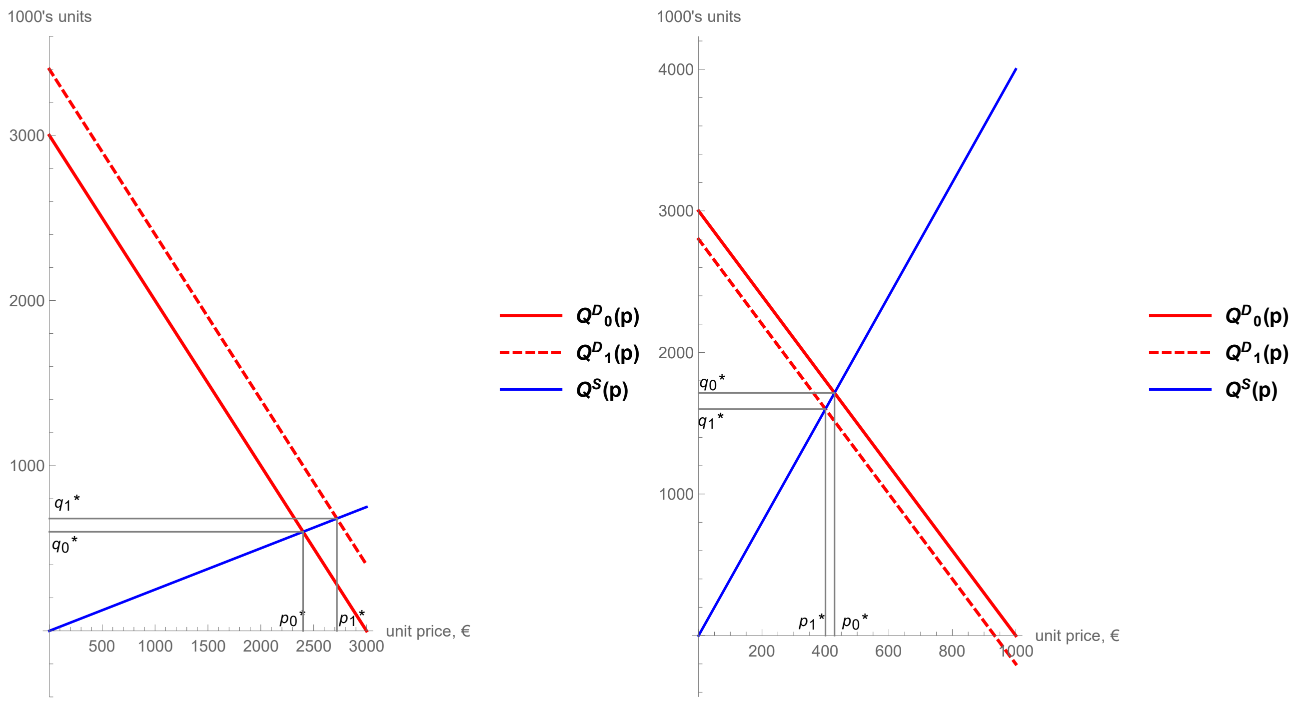
(c) First note that the unfortunate passing of an owner decreases the surplus by half the valuation of the owner. We don't have to take this into account as subsequent trading is considered only. Optimally the book will be allocated to the person with highest valuation of those not holding a book. This valuation is 95. In the most fortunate case the bookholder who dies has a valuation of 100 and thus her son's valuation is 50. Therefore the maximum surplus generated by such trade is $95 - 50 = 45$.

4. In 2018, there were advanced political plans to place a lump sum subsidy for electric bike purchases. Upon purchasing an e-bike, the owner would apply for the subsidy and receive the payment herself if the application was approved.

Should the plan have materialized, it would likely have caused an upward shift in the demand for e-bikes, as depicted in Figure 4a ($Q_0^D(p) \to Q_1^D(p)$). A subsidy affects relative prices of e-bikes and any other goods, e-bikes becoming relatively cheaper. Thus it is likely that consumers shift some of their consumption from other forms of transportation, recreation, whatever, to e-bikes.

In the new equilibrium both quantity and price are increased. Supply curve is unaffected by the subsidy as the subsidy is given directly to the buyer. Note that as the demand for e-bike in this example is quite inelastic, i.e. quantity reacts only little to price changes, subsidy would have relatively little effect on quantity while strongly boosting the price.

E-bikes are likely to be substitutes for non-electric bikes. Once e-bikes become relatively cheaper, it's likely that the demand for non-electric bikes shifts down, which is depicted in Figure 4b ($Q_0^D(p) \to Q_1^D(p)$). Quite naturally, this shift would have opposite effect on price and quantity in the new equilibrium from that in the e-bike market. In this particular exam-



(a) The market for electric bicycles. **(b)** The market for non-electric bicycles.

Figure 4: Two intertwined markets.

ple the demand for normal bikes is elastic, at least compared to that of e-bikes. Therefore the drop in quantity is quite sizable relative to the magnitude of the shift itself.

The goods in these markets are substitutes. Spare parts for e-bikes would've been a good candidate to illustrate complementarities. In that case, the effects on demand and therefore equilibrium price and quantity in the spare parts market would've been exactly the opposite than in the traditional bike market.

Naturally, these two markets wouldn't be the only ones affected by such subsidy. The market for automobiles comes first into mind and must have occurred to legislators as well. Although not as obviously as traditional bikes, automobiles might act as substitutes for e-bikes and therefore the subsidy could have similar effect on that market as it does on the market for traditional bikes.

In the later stages of your studies (at least for those with Economics as a major) you'll study general equilibrium theory. By that time, if not before, it'll become obvious that any perturbances on one market sends trembles across the whole economy. In this exercise, we only touch on that idea however.

5. (a) In (a) and (b) quantity is in liters/month and price in euros per liter.

   The (price) elasticity of demand desrcibes how sensitive the quantity demanded is to price changes. That is

$$elasticity\ of\ demand = \frac{relative\ change\ in\ quantity\ demanded}{relative\ change\ in\ price} \iff \epsilon^D = \frac{dQ^D/Q^D}{dp/p} \tag{1}$$

We will use the same discrete approximation as in the lectures and use the middle point between the quantities and prices. Our formula becomes $\epsilon^D = \frac{dQ^D/\bar{Q}^D}{dp/\bar{p}} = \frac{dQ^D/\frac{Q_1^D+Q_0^D}{2}}{dp/\frac{p_1+p_0}{2}}$. Plugging in values, the elasticity becomes $\frac{\frac{11000-10000}{(11000+10000)/2}}{\frac{15-16}{(16+15)/2}} = -\frac{31}{21} \approx -1.48$. That is, if price increases (decreases) by one percent, demand decreases (increases) 1.48 percent. This does not have to be good estimate globally but only in this particular neighborhood.

(b) In the equilibrium, quantity demanded and supplied are the same and therefore we know also that $Q^D = 9500$ in the equilibrium with restrictions imposed. In our framework, restrictions will cause an upward shift of the supply curve.

Our estimate for the elasticity from (a) is our only description how demand reacts to price changes. Let's plug it into 1 along with anticipated new production: $-\frac{31}{21} = \frac{\frac{9500-10000}{(10000+9500)/2}}{dp/p} \implies dp/p = \frac{14}{403} \approx 3.47\% \implies p_1 = 16 \times (1 + \frac{14}{403}) \approx 16.56$. That is, price would increase circa 3.47% in reaction to the shock.

Total revenue is simply quantity sold times price, $p_t \times Q_t^D,\ t = 0, 1$, and that's bound to decrease from 160 000 euros/month to 157 280 euros/month. The reduction isn't very sizable as increased price partly compensates the loss in volume.

(c) Relative change in revenue is given by $\frac{dR}{R} = \frac{dp}{p}(1 + \epsilon^D)$. For intuition, the last term in the parenthesis captures the effect of price change on quantity, the first term the effect on price which naturally is one-to-one. Then we can derive the requested measures in a straightforward manner:
Argentine: $0.04(1 - 1.5) = -2\%$
Belgium: $0.04(1 - 0.7) = 1.2\%$
Canada: $0.04(1 - 1) = 0\%$

(d) Now we have to aggregate supply from 1000 identical producers: $Q^S(p) = \sum_{i=1}^{2000} q_i^S(p) = 2000 \times (p/200 - q/25) = 10p - 80$. As previously, we find the equilibrium by setting quantity supplied equal to quantity demanded: $Q^D(p) = Q^S(p) \iff 100 - 2p = 10p - 80 \implies p^* = 15$
Substitute p^* into demand curve for equilibrium quantity:
$Q^D(p^*) = 100 - 2 \times 15 \implies Q^* = 70$.
As probably expected, having double the supply at any level of p increases quantity and decreases price.

(e) I supply these model solutions completely kudos-inelastically meaning that you can be sure to have your weekly solution manual irrespective of amount kudos, or feedback I receive. Quality of the model solutions may be affected by the feedback which in turn affects the demand, but that is a different story

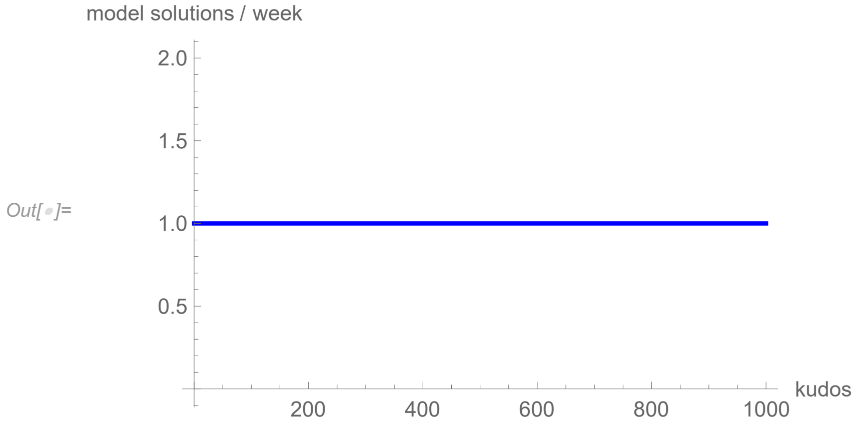


Figure 5: Supply of model solutions

6. In contrast to competitive markets, we don't assume all the agents being price-takers here. In this case we have a monopoly who takes into account the fact that supplying more requires lowering the price of all the units sold. Monopoly effectively is picking the point on the demand curve it fancies the most.

   The monopolist's objective is to choose quantities (or prices, as one determines another) at which marginal revenue, $MR(q)$, equals marginal cost, $MC(q)$. Assume the opposite: if marginal cost is below marginal revenue it makes sense to supply more as the associated costs are surpassed by the revenue. Vice versa, monopolist should cut production if it's producing units that give less revenue than they bring costs.

   The monopolist's total revenue is simply quantity times the price she can charge for that quantity, $TR(q) = q \times p^D(q)$. Marginal revenue is the rate of change of total revenue wrt quantity, $MR(q) = \frac{\partial \mathrm{TR}(q)}{\partial q}$.

   (a) In this exercise, geography comes into help of the monopolist, allowing it to discriminate between south and north.

   Begin with inversing the demand curves: $Q_S^D(p) = 18 - p/2 \iff p_S^D(q) = 36 - 2q$ for south and $p_N^D(q) = 72 - 3q$ analogously for north. Then the total revenues become $TR_S(q) = q(36 - 2q) = 36q - 2q^2$ and $TR_N(q) = q(72 - 3q) = 72q - 3q^2$. Taking derivatives with respect to q yields marginal revenues: $MR_S(q) = 36 - 4q$ and $MR_N(q) = 72 - 6q$.

   Marginal cost is constant and the same for both markets, $MC_S(q) = MC_N(q) = 10$. Therefore the quantity chosen on one market doesn't affect the marginal cost on the other and we can treat these markets in complete separation. Optimal quantities are then given by $MR_S(q) = MC_S(q) \iff 36 - 4q = 10 \implies q_S^* = 13/2$ and $MR_N(q) = MC_N(q) \iff 72 - 6q = 10 \implies q_N^* = 31/3$.

   Plugging these into the respective demand curves gives us the optimal prices for each market: $p_S^* = p_S^D(q_S^*) = 36 - 2 \times 13/2 = 23$ and $p_N^* = p_N^D(q_N^*) = 72 - 3 \times 31/3 = 41$.

Consumer surplus is the sum of the differences between the price paid and consumer's willingness to pay. Geometrically, this is the area between inverse demand curve and price. For south, this is given by $CS_S = ((36-23)\times(13/2-0))/2 = 169/4 \approx 42$, $CS_N = ((72-41)\times(31/3-0))/2 = 961/6 \approx 160$ for north.

(b) Now the monopolist has to decide one price and stick to it in both regions. Cranking up (down) the uniform price will detract (attract) buyers on both markets. How demand depends on price is thus described by $Q^D(p) = Q^D_S(p) + Q^D_N(p) = 42 - \frac{5}{6}p$.

We know that monopolist makes a profit $p-10$ for each unit she sells while demand curve gives the volume of sales at any given price. Then the profits are given by $\pi(p) = (p-10)(42-\frac{5}{6}p) = 50\frac{1}{3}p - \frac{5}{6}p^2 - 420$. First order condition (FOC, i.e. take first derivative and find its root) becomes $1/3(151-5p) = 0 \implies p^* = 151/5 = 30\frac{1}{5}$. Plugging this into regional demand curves we get optimal quantities: $q_S^* = 18-(151/5)/2 = 29/10$ and $q_N^* = 24-(151/5)/3 = 209/15$.

Profits under regulation are given by $\pi_R^* = (151/5-10)(42-151/5\times\frac{5}{6}) \approx 340$. Unregulated profits are $\pi_U^* = (23-10)(18-23/2)+(41-10)(24-41/3) \approx 405$. Profits decrease. Note that by selling just in the north the monopolist could generate a profit of circa 320. South becomes quite irrelevant for the business after regulation.

Consumer surpluses under regulation are $CS_{SR}^* = ((36-151/5)*(29/10-0))/2 = 841/100 = 8.41$ and $CS_{SR}^* = ((72-151/5)*(209/15-0))/2 \approx 291$. Total consumer surplus increases.

(c) Instead of constant marginal cost we now have increasing marginal costs in total quantity, $MC(q) = 2+0.5(q_S+q_N)$. Note that now the quantity choices on the markets are intertwined. Yet the familiar equivalence applies, only with a slight modification: $MR_S(q_i) = MC(q_S+q_N)$, $i = S, N$. So the following have to hold simultaneously:

$$36 - 4q_S = 2 + 0.5(q_S + q_N)$$
$$72 - 6q_N = 2 + 0.5(q_S + q_N)$$

LHS's are equal, $36-4q_S = 72-6q_N \iff q_S = \frac{3}{2}q_N - 9$. Plugging this into second equation yields $q_N = 298/29 \implies q_S = 186/29$. Prices are then $p_S^* = p_S(q_S^*) = 36-2\times186/29 = 672/29 \approx 23.2$ and $p_N^* = p_N(q_N^*) = 72-3\times298/29 = 1194/29 \approx 41.2$. Profits are obtained by subtracting total costs (TC^*) from total revenue ($TR_i^* = q_i^* p_i^*$). Total cost, geometrically a trapezoid, is given by $TC^* = (2+(2+(q_S^*+q_N^*)/2))/2\times(q_S^*+q_N^*)$. Profits in the optimum therefore are $\pi^* = TR_S^* + TR_N^* - TC^* = 13592/29 \approx 469$. Note that reporting profit by region doesn't really make sense as the costs are shared overhead, i.e. intertwined and cannot really be factored by area.

7. (a) At price $p < 5$ all the 3000 potential buyers will purchase the mask, none at price $p > 35$. Because the distribution of reservation price is uniform between these limits, demand

curve will be linear: new buyers are attracted at constant rate as price increases. Knowing two points on a line we can deduce it's equation: $Q^D(p) = 3500 - 100p$ if $5 < p < 35$, $Q^D(p) = 3000$ if $p \leq 5$, $Q^D(p) = 0$ if $p \geq 35$, as depicted in Figure 6.

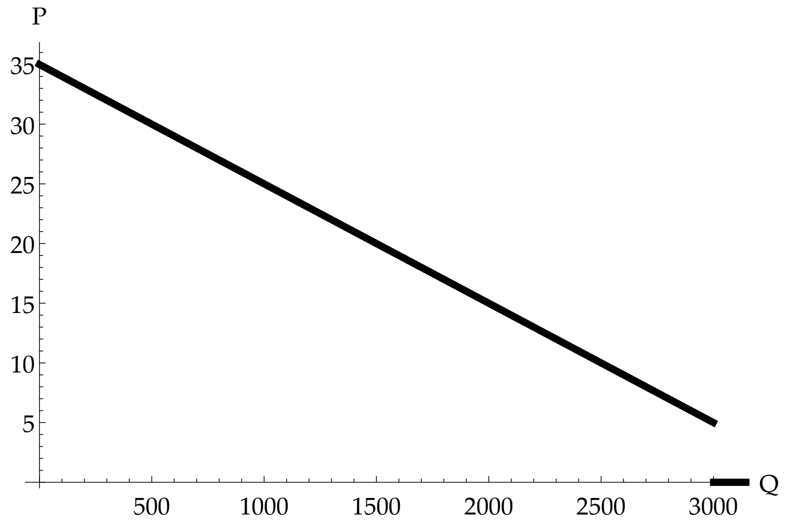


Figure 6: Demand for masks.

(b) Our demand curve is defined piecewise so we have first to make sure we're dealing with the correct part of the curve. Clearly price will be at least five because anything lower wouldn't increase sales. It must be less than 35 as well as the monopoly will sell nothing at such price. Therefore our effective demand curve at the optimum must be $Q^D(p) = 3500 - 100p$ given there's production at all. This seems quite trivial and admittedly it is in this case. However, sometimes this saves you from unpleasant surprises.

Monopoly will produce quantity such that marginal cost $MC(q) = x$ equals marginal revenue $MR(q) = \frac{\partial}{\partial q} qp^D(q)$. Inverse demand is obtained as $Q^D(p) = 3500 - 100p \iff p^D(q) = 35 - q/100$. Marginal revenue becomes $MR(q) = \frac{\partial}{\partial q} q(35 - q/100) = 35 - q/50$. Setting $MR(q) = MC(q)$ yields $35 - q/50 = x \implies q^* = 1750 - 50x$. Plugging this into the demand curve yields $p^* = p^D(q^*) = 35 - (1750 - 50x)/100 = \frac{35+x}{2}$.

As the club must pay the upfront marketing cost, that should also be covered by sales. That is, profits must be positive $\pi(q^*) = (p^* - x)q^* - 10000 = (\frac{35+x}{2} - x)(1750 - 50x) - 10000 \geq 0 \implies x \leq 15.$[1]

With $x > 15$ production cannot be made profitable. The club would produce zero quantity and optimal price is indeterminate. Note that minimum order size isn't binding here: optimal supply would be $q^* = 1750 - 50 \times 15 = 1000$ in the limiting case $x = 15$ and more in other plausible cases. Therefore the club will always produce more than 500 masks if it produces any.

[1]If unsure, what is happening in the last equivalence, try the following. Find the roots of the profit polynomial. Inspect its curvature and determine, which points lie above zero.

As argued above, at $x = 5$ production would still be profitable and thus $p^* = p^D(q^*) = \frac{35+5}{2} = 20$.

(c) As per (b), in either of the scenarios cost is low enough $5 < 10 < 15$ for the production to be made profitable, so the club will incur the marketing cost. The question about optimal price remains.

Denote the pre-committed price by $\bar{p}$. The club will sell to anyone willing to buy at that price given the price exceeds marginal costs. The number of such buyers is $Q^D(\bar{p}) = 3500 - 100\bar{p}$.

At the stage when production is decided the club cannot play with the price anymore. Usually cutting production would allow monopolist to charge higher prices, but this time she has only to lose from rationing the supply: only quantity would decrease with price remaining unchanged. Therefore, when deciding on $\bar{p}$ the club effectively decides quantity produced as well.

In one out of four cases we will have a constant marginal cost $MC_H = 10$, otherwise the cost will be $MC_L = 5$. Therefore the expected profits before observing the costs will be $E[\pi(\bar{p})] = \frac{1}{4}(\bar{p} - 10)(3500 - 100\bar{p}) + \frac{3}{4}(\bar{p} - 5)(3500 - 100\bar{p}) - 10000 = (\bar{p} + 25/4)(3500 - 100\bar{p}) = 3500\bar{p} - 100\bar{p}^2 + 25/4 \times 3500 - 2500\bar{p}/4$.

Taking the first order condition, that is, setting first derivative of profits wrt $\bar{p}$ to zero, yields $4125 - 200\bar{p} = 0 \implies \bar{p}^* = 20\frac{5}{8}$. Clearly this is above the marginal costs in either scenario. As profit function is a downward opening parabola (second order term has negative multiplier) with negative second derivative (second order condition, SOC), profit indeed reaches its a maximum at this point.

8. All the quantities are in thousands of liters a month, prices in marks per thousand liters.

Start with aggregating the demand by adding each individual household's demand on top of each other: $Q^D(p) = \sum_{i=1}^{1000} Q_i^D(p) = \sum_{i=1}^{1000} 10 - p = 1000(10 - p) = 10000 - 1000p$.

In the inverse form the demand is $P^D(q) = 10 - q/1000$.

(a) Our familiar condition of matching marginal revenue and cost applies: $MC(q) = MR(q)$ in the monopolist's optimum. $MC(q) = 1$ and $MR(q) = \frac{\partial q(10-q/1000)}{\partial q} = 10 - q/500$. Setting these equal yields $10 - q/500 = 1 \implies q^* = 4500$. There are 1000 households and therefore consumption per household is 4.5.

Plugging optimal quantity into inverse demand we get $p^* = p^D(q^*) = 10 - 4500/1000 = 5.5$. Waterwork will make monthly positive profits as $\pi(q^*) = (5.5 - 1) \times 4500 - 3000 = 17250$. Otherwise it would optimally have run the plant down.

Demand curve meets marginal costs at $1 = 10 - q/1000 \implies q^* = 9000$. Therefore deadweight loss is given by $(9000 - 4500)(5.5 - 1)/2 = 10125$ marks a month.

Consumer surplus is $(10 - 5.5)(4500 - 0)/2 = 10125$ and this divided between 1000 households gives surplus 10.125 euros per month each.

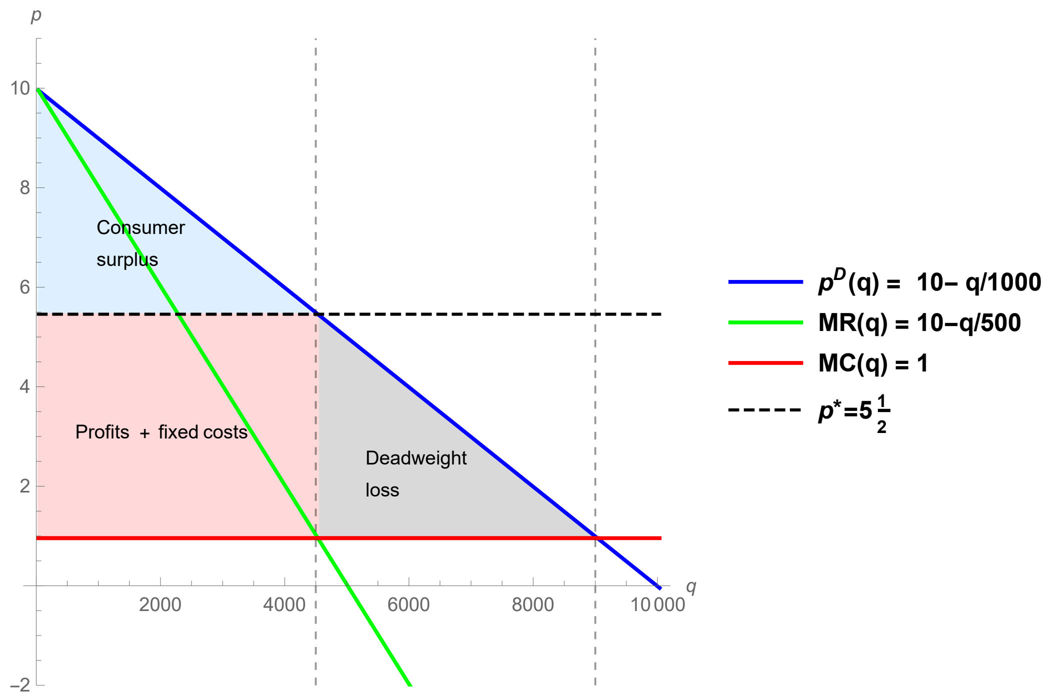


Figure 7: Monopolist waterwork in (a).

(b) Setting $p = 0$ generates the greatest consumer surplus, $(10000 - 0)(10 - 0)/2 = 50000$, which makes 50 per household a month. Profits would be $\pi(q^0) = (0 - 1) \times 10000 - 3000 = -13000$.

Producing the last 1000 units is wasteful as the cost exceeds consumers' willingness to pay. Deadweight loss (that little triangle in the lower right corner of Figure 7) is thus $(10000 - 9000)(1 - 0)/2 = 500$.

(c) Consumer surplus is decreasing in price. Therefore we must find the greatest quantity, or equivalently lowest price, at which the waterwork can earn at least zero profits. Profits are zero when price equals average cost. Average cost is $\mathrm{AC}(q) = \mathrm{TC}(q)/q = 3000/q + 1$. Let's find the level of production at which consumers are willing to pay the average cost price.

$$\begin{aligned} P^d(q) = \mathrm{AC}(q) &\Longleftrightarrow 10 - q/1000 = 3000/q + 1 \\ &\Longrightarrow 10q - q^2 - 3000 - q = 0 \end{aligned}$$

This is a second degree polynomial, with two roots. Since consumer welfare is increasing and average cost is decreasing in quantity q, the larger root is the sensible one here. Therefore $q^{**} = 500(9 + \sqrt{69}) \approx 8653$ and $p^{**} = P^d(q^{**}) = 10 - q^{**}/1000 \approx 1.35$.

Total consumer surplus and deadweight loss can be calculated as simple areas, similarly as in Figure 7. Deadweight loss is given by $(9000 - 500(9 + \sqrt{69}))(.5(11 - \sqrt{69}) - 1)/2 \approx 60.1$. Total consumer surplus stands as $(500(9 + \sqrt{69}) - 0)(10 - .5(11 - \sqrt{69}))/2 \approx 37440$

a month, which makes 37.44 for each household. After taking into account fixed costs, profits are zero (by construction).

(d) Through similar aggregation as before, the new demand curve is $Q_2^D(p) = \sum_{i=1}^{500} Q_i^D(p) = \sum_{i=1}^{500}(10-p) = 500(10-p) = 5000 - 500p$. Its inverse stands as $P_2^D(q) = 10 - q/500$. Following the same steps as in (a), we have $MC(q) = MR(q) \iff 1 = \frac{\partial}{\partial q} q(10 - q/500) = 10 - q/250 \implies q_2^* = 2250$. Consumption per household remains the same, 4.5. Plugging into inverse demand curve we get $p_2^* = P_2^D(q_2^*) = 10 - 2250/500 = 5.5$. Because households are identical and buy same amount of water at the same price, consumer surplus must be unchanged as well.

The waterwork would make *profits* $(5.5-1) \times 2250 - 3000 = 7125$. Marginal costs meet demand at $1 = 10 - q/500$ from which we can solve the efficient level $q = 4500$. Therefore *deadweight loss* is $(4500 - 2250)(5.5 - 1)/2 = 5062.5$ marks per month, through similar geometry as before.

As for (c), average cost pricing condition is now $10 - q/500 = 3000/q + 1$, for which the reasonable root is now $q_2^{**} \approx 4137$, resulting in a price of $p_2^{**} \approx 1.73$, and consumption per household of about 8.3 thousand liters per month. Using similar geometry as before, consumer surplus per household is $17119/500 \approx 34$ marks per month, and profits are zero by construction.

After the population decline, the fixed cost of the waterworks infrastructure has to be spread over a smaller number of consumers. The price must increase and average consumer welfare must decrease. Notice that the direction of the change in total surplus (consumer surplus plus profit) does not depend on the pricing regime.

9. Common defense in Northland is arguably a public good. You exclude any clans from it. At least during peacetime, defense is more of a threat that won't exhaust, making it a public good. A threat on eastern border probably won't scare an enemy coming from the west and so it's probably not purely a public good. Let's, however, assume that fighter planes are mobile enough for us to ignore such considerations.

All the monetary quantities are in millions of euros, defense quantities in number of fighter planes

(a) Northland should purchase a number of fighter planes such that the marginal benefit from the last plane purchased equals the associated cost.

As a reminder, inverse demand depicts the consumers' (clans' in this case) willingness to pay for the q:th unit of the good. Inverse demands for the clans are given by

$Q_A(p) = 60 - 6p \iff p_A(q) = 10 - q/6$
$Q_B(p) = 80 - 5p \iff p_B(q) = 16 - q/5$
$Q_C(p) = 50 - 2p \iff p_C(q) = 25 - q/2$

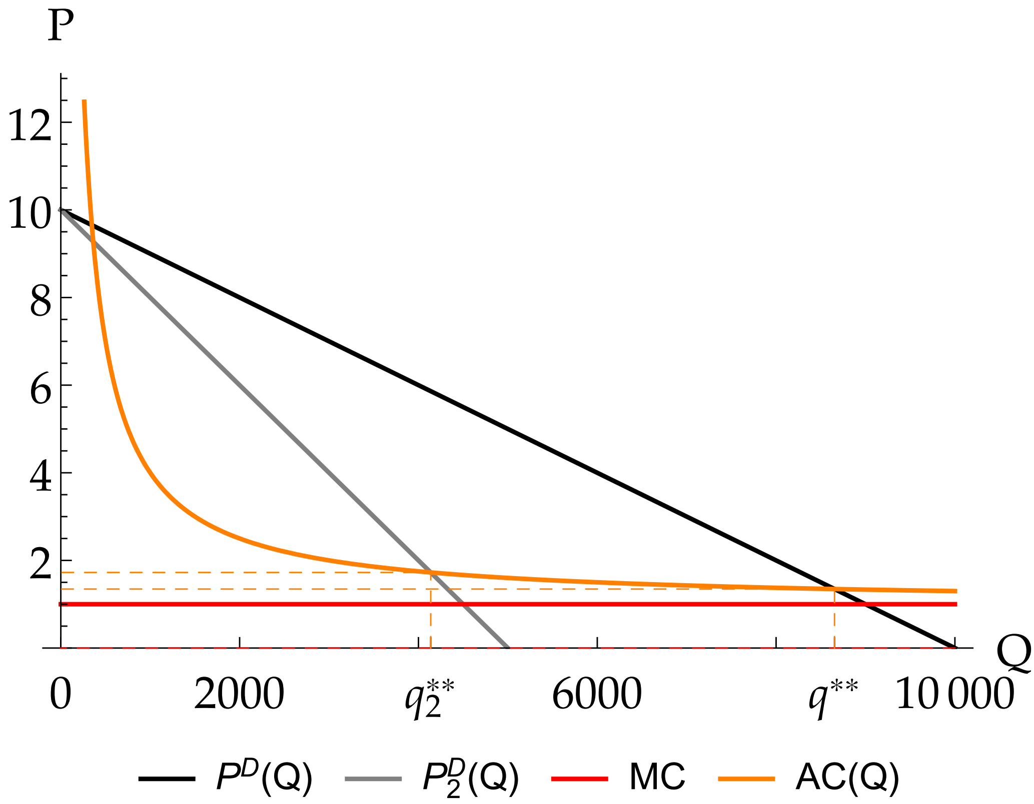


Figure 8: Average cost pricing before and after the population decline. Choosing the smaller root of $\mathrm{AC}(q) = P^d(q)$ would correspond to choosing the first crossing of the two curves; moving to the right along the q-axes from it adds consumption that is valued at above marginal cost and average costs.

Marginal cost is constant, $MC(q) = MC = 25$. Summing up the above equations we get the marginal benefit and efficient quantity thus is such that $25 = 10 - q/6 + (16 - q/5) + (25 - q/2) \implies q^* = 30$. We also must ensure that no clan's willingness to pay isn't negative at the efficient level: $p_A(q^*) = 5 > 0,\ p_B(q^*) = 10 > 0,\ p_C(q^*) = 10 > 0$.

(b) The constitution obliges the clans to share the cost evenly and therefore the burden is $MC \times q^*/3 = 25 \times 30/3 = 250$ for each of the clans. Applying some geometry to Figure 9, we get
$TS_A = (5-0)(30-0) + (10-5)(30-0)/2 - 250 = -25$
$TS_B = (10-0)(30-0) + (16-10)(30-0)/2 - 250 = 140$
$TS_C = (10-0)(30-0) + (25-10)(30-0)/2 - 250 = 275$

(c) Each clan faces a marginal cost $MC = 25/3$ which would optimally match marginal benefit for that clan. Should they be able to have their will, we'd have the following spendings:
$p_A(q) = 10 - q/6 = 25/3 \implies q_A^* = 10$
$p_B(q) = 16 - q/5 = 25/3 \implies q_B = 38\frac{1}{3}$

$p_C(q) = 25 - q/2 = 25/3 \implies q_C = 33\frac{1}{3}$
Each of the inverse demand curves is of form $p_i(q) = a_i - b_i q$. Therefore the total surplus before costs is a nice trapezoid (with left side of a_i and right side of $a_i - b_i q$) whereas costs are even nicer rectangle, yielding
$TS_i(q) = q(a_i + a_i - b_i q)/2 - 25q/3 = q(a_i - 25/3) - b_i q^2/2.$
These are downward opening parabolas which are symmetric. Therefore, the further we're from the optimal q, the greater is the offset from maximal total surplus and also $q_B^* = 38$ and $q_C^* = 33$, as planes don't come in fractions. By the same argument we know that for A, $q_A^* \succ q_C^* \succ q_B^*$, for B $q_B^* \succ q_C^* \succ q_A^*$ and for C $q_C^* \succ q_B^* \succ q_A^*$.

C is the median voter. Because surpluses are single-peaked, we know that Arrow's impossibility theorem won't apply and *median voter will prevail.*

To grasp the intuition, in pairwise votes A's proposal never gets majority of the votes as it's the worst option for other chiefs. If A is voted on the first round, it's eliminated and second round will be B vs C which C will take. If A is not removed, first round must have been B vs C. C will prevail and beat A on the second round.

Why single-peakedness is important is best explained by counterexample: assume B's preferences would be B $q_B^* \succ q_A^* \succ q_C^*$ and others' unchanged. Then A vs B would have A as the winner, A vs C C as the winner and B vs C B as the winner. For example, voting first B vs C and then B vs A would have A chosen, voting first A vs B and then A vs C would have C as the result.

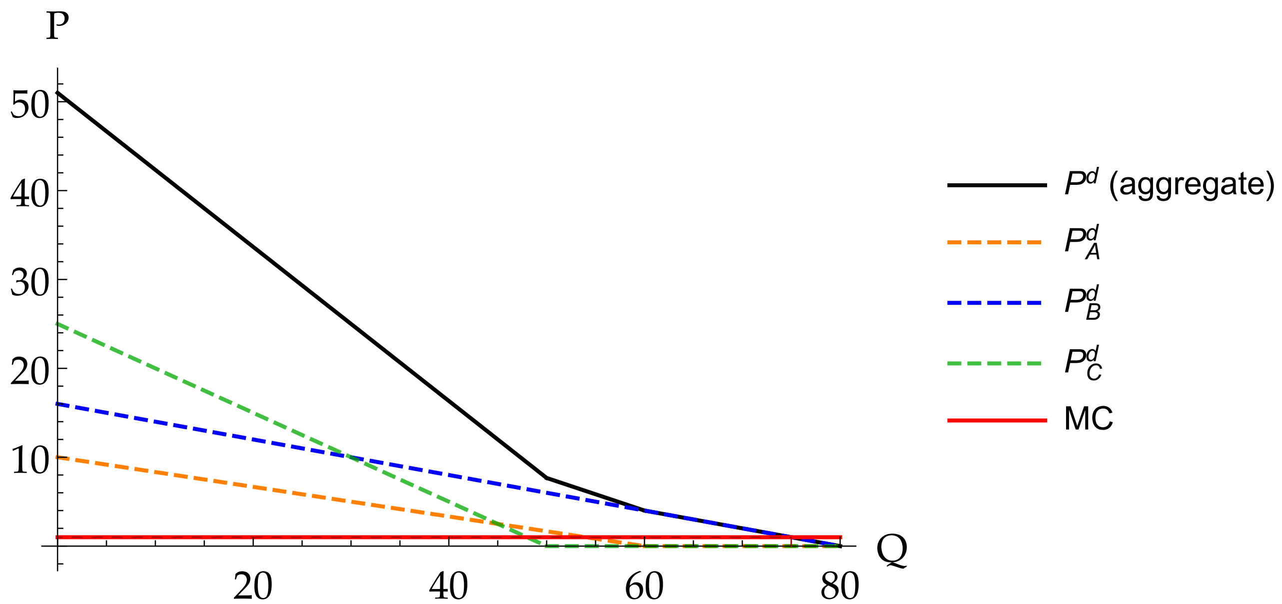


Figure 9: Demand for the public good by each clan separately and the aggregate for Northland.

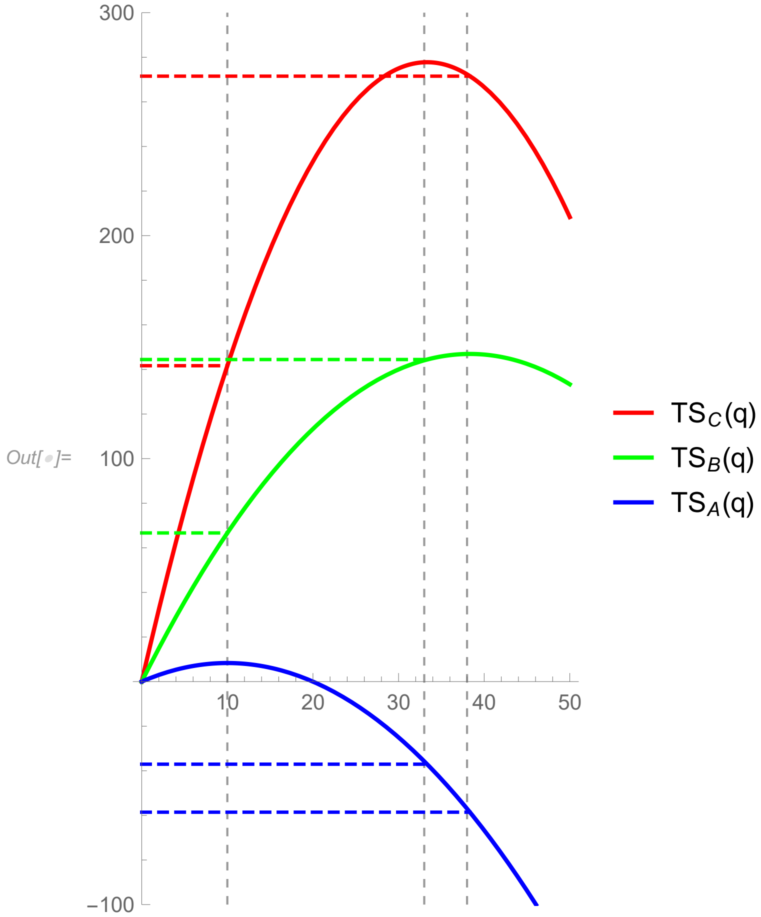


Figure 10: Surpluses of the clans as functions of quantity.

10. i Laptops. When one person is using a laptop, nobody else can. It is a physical good so it is simple for sellers to restrict access to only those who pay.

    ii Ad-free music streaming services. Thanks to encryption, consumers have to pay to be able to listen, but no matter how much one listens to it does not reduce other's ability to consume the same music.

    iii Fresh air these days. The more people there are around in public spaces means that others' ability to enjoy breathing there is diminished, because of the risk of contracting Covid-19. Yet it is not possible to exclude others from entering public spaces.

    iv National weather forecasts in Finland. They are distributed freely by the Finnish Meteorological Institute, and as for all information goods, one person's consumption of it does not diminish the amount left for others.

11. Reservation price is the lowest price at which the company is willing to supply anything at all. It will make zero profits at reservation price, and would make deficit at lower prices should it operate.

    All the prices are in thousands of euros.

(a) If profits are zero, then it must be that unit price equals average cost: $p = TC(1000)/1000 = (500000 + 100000)/1000 = 600$.

(b) We used a shortcut above, but we could've equally well set profits to zero and solve for price. That is what we do here, with the difference that now the profits are expected: $E[\pi(p)] = (p - 100) \times 1000 + (1/2)(200 - 100) \times 2000 - 500000 = 0 \iff p = 500$.

When calculating expected profits we first figure out, what are all the possible, mutually exclusive states of the world (order, no order), calculate the profits in each state and multiply these profits by the corresponding probabilities (0.5, 0.5). If some cost, for example, is to be paid in any state, we don't have to calculate it into profits in each individual state as here is done with leasing costs.

At reservation price, firm will make profit (deficit) if the order for B gadgets materializes (is canceled), but zero on expectation.

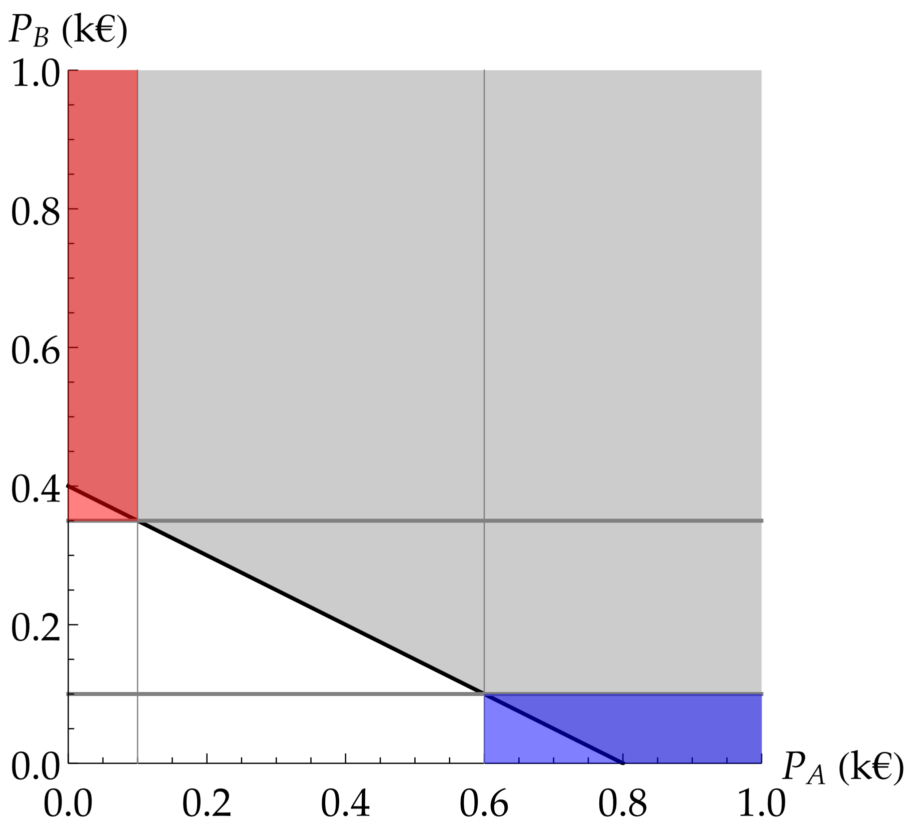


Figure 11: Both deals would be accepted when unit prices $\{p_A, p_B\}$ are in the gray region. In the blue region only A and in the red region only B is accepted.

(c) Gadget's price must always exceed or equal marginal cost to be produced, $p_i \geq 100$. Firm will naturally produce a gadget at full capacity if it produces that gadget at all.

Only one of the gadgets is produced when one gadget's price is below that threshold but other gadget's price is high enough alone fixed costs to be covered by sales of that gadget only.

Non-negative profits condition is, as per (a), given by $p_A \geq 600$ when $p_B < 100$. Only A is produced with these values.

Condition becomes $\pi(p_A, p_B) = -500000 + (p_B - 100) \times 2000 = 0 \iff p_B \geq 350$ when $p_A < 100$. Only B is produced then.

If $p_A,\ p_B \geq 100$ condition is written as $\pi(p_A, p_B) = -500000 + (p_A - 100) \times 1000 + (p_B - 100) \times 2000 = 0 \iff p_A + 2p_B \geq 700$ and both gadgets are produced. Figure 11 shows the regions in price space where different acceptance decisions are optimal.

12. All prices are in thousands of euros.

(a) The company has the following decisions at hand: (i) whether to develop blueprints, (ii) which quality or qualities of prototypes to produce (iii) whether to build a plant and (iv) how many robots to supply.

To shave off a couple of branches from our decision tree, observe that in case of failed certification no units will be sold or supplied and neither will the plant be built. Non-friendly robots can only be sold at a price below marginal costs which makes them completely irrelevant.

If the company develop a blueprint, it won't sell any robots without certification, which cannot be obtained without a prototype which therefore will always be built along with a blueprint. Therefore (i-ii) are partially hand in hand.

The tree has been simplified to take into account that, once the prototype passes the certification, the company will always build the plant. At that point the development and prototype costs have already been sunk and (as we will soon see) it's possible to sell human-friendly robots at a profit. Basically stages (iii-iv) are incorporated in the payoff realizing in the end of each branch

In (iv), past actions (i-iii) will be sunk costs and the company will choose optimal quantity and price irrespectively of those. The company is our usual monopoly and will set marginal revenue equal to marginal cost. Inverse demand is given by $p^D(q) = 50 - q/2000$, and marginal revenue $MR(q) = \frac{\partial}{\partial q} q p^D(q) = 50 - q/1000$. $MC(q) = MR(q) \iff 50 - q/1000 = 25 \implies q^* = 25000$. Therefore capacity constraint does not bind. Optimal price is $p^* = p^D(q^*) = 37.5$ and profits before any fixed costs $\pi^* = 25000(37.5 - 25) = 312500$.

At stage (iii) plant building costs are not yet sunk, and as $200000 < 312500$, the plant will be built if certification is passed as argued before. Therefore at (ii) the company will know that a successful (failed) prototype will give a profit of 112500 (0) before sunk costs so far, that is, the costs of developing the blueprint.

The firm has two shots: it can either start with building a cheap and more risky low quality prototype, or vice versa. If developing a high quality one first and failing, the company knows that low quality one will fail as well.

Assume firm begins with a low quality one. With probability 0.2 it succeeds. If it doesn't succeed, the probability that high quality prototype succeeds is 0.5. To see this, assume there are 100 possible states of the world. In every state where low quality prototype succeeds also high quality prototype succeeds. If low quality prototype fails, we can exclude those 20 successful states in each of which also high quality prototype succeeds. The remaining 80 states have 40 successful and 40 failed high quality prototypes.

The expected profits (before sunk costs thus far) are given by $0.2(-15000+112500)+(1-0.2)\times 0.5(-(15000+35000)+112500)+(1-0.2)(1-0.5)(-15000-35000) = 24500$. That is, with probability 0.2 firms succeeds on first try, with complementary probability it has to retry, which succeeds with probability 0.5. If neither prototype succeeds, firm just has to pay costs.

Starting with a high quality one gives $0.6(-35000+112500)+(1-0.6)(-35000) = 32500$. If high quality fails, it's not worth experimenting with a low quality one as it will fail. Trying out only low quality option would yield expected profits of 7500, through similar arithmetic.

Therefore the company would optimally try its luck with the high quality prototype. Blueprints cost 20000 and the company can expect profits of 32500 if it takes optimal actions in (ii-iv). The optimal decision is to develop the blueprints in (i), as it generates expected profit of $E[\pi^*] = 12500$.

Note that we could've extended the decision tree in Figure 12 by adding nodes (iii) and (iv) after failed and passed certifications. Because the optimal actions beyond (ii) are quite trivial and basic monopoly optimization, these branches were omitted despite being parts of the decision.

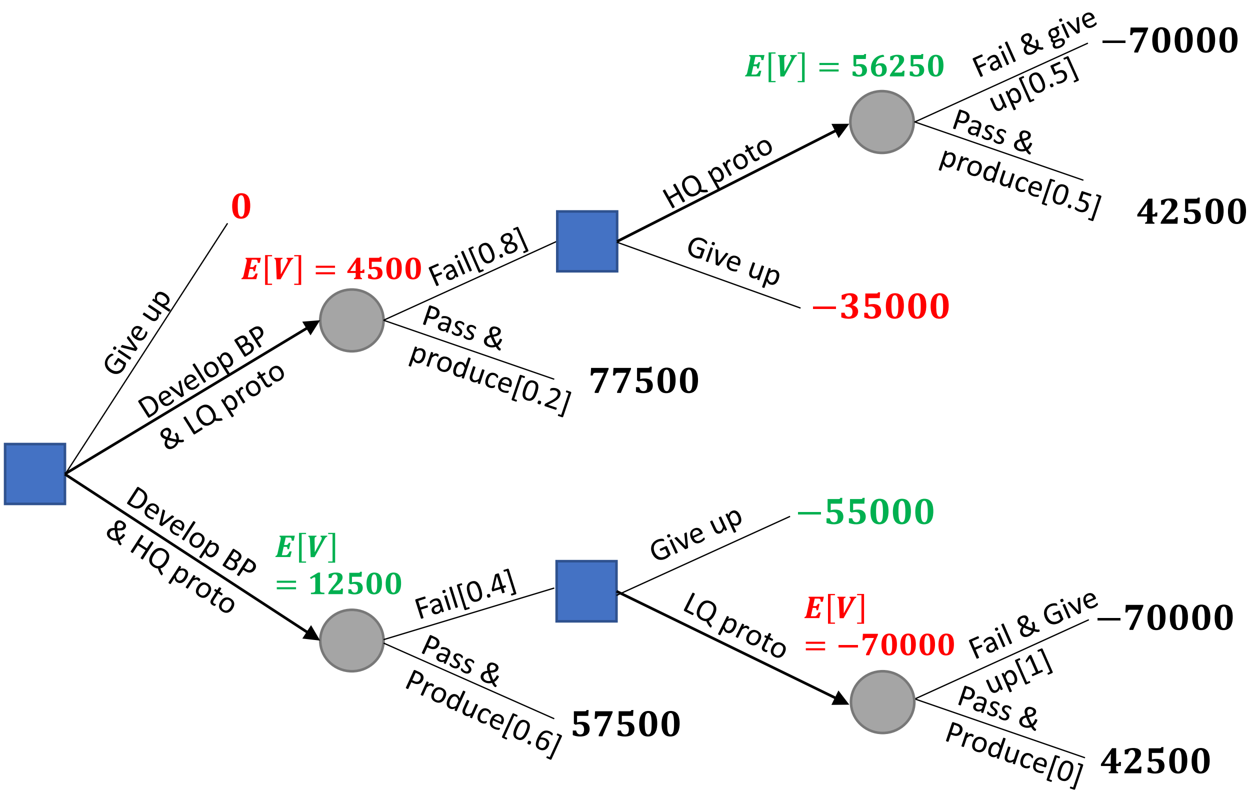


Figure 12: Decision tree in 12a.

(b) Assume that firm would see the outcomes by itself without consulting. At the first stage, the company would speculate which certification outcomes it'd see on the second stage. There would be a probability of 0.2 to get away with just a low quality prototype, probability of $0.6 - 0.2 = 0.4$ of getting the certification with high quality prototype (as the firm won't develop a high quality one if low type would be successful) and probability of $1 - 0.2 - 0.4 = 0.4$ not getting the certification at all.

Expected profits from making the blueprint would be $-20000 + 0.2 \times (112500 - 15000) + 0.4 \times (112500 - 35000) + 0.4 \times 0 = 30500$. Company would make 12500 profits following the optimal path in the absence of the consulting service. Therefore the reservation price for the services would be $30500 - 12500 = 18000$.

(c) Adjusting the number of buyers will affect the optimal solution in (iv) and potentially send trembles backwards. Inverse demand becomes $p^D(q, N) = 50 - q/2N$, marginal revenue $MR(q, N) = 50q + q/N$ and optimality condition $50q - q/N = 25 \implies q^* = 25N \implies p^* = 50 - 25/2 = 37.5$.

Profits before sunk costs at stage (iv) become $25N(37.5 - 25) = 312.5N$. Counting in stage (iii) costs we have $112.5N$. Remember that node (iii) was never going to be decisive.

On the second stage we'll have expected profits before sunk costs $0.2(-15000 + N \times 112\ 1/2) + (1 - 0.2) \times 0.5(-(15000 + 35000) + 112.5N) + (1 - 0.2)(1 - 0.5)(-15000 - 35000) = 67.5N - 43000$ if starting with a low type one, $0.6(-35000 + 112.5N) + (1 - 0.6)(-35000) = 67.5N - 35000$ if checking high quality prototype first. It's thus always profitable to start with a high quality one if it's worth experimenting or developing the blueprints at all. The break-even point for developing blueprints is $67.5N - 35000 - 20000 = 0 \iff N = 22000/27 \approx 814.2$. If N is above the break-even point then the firm will develop the blueprints and follow the same optimal path as in (a), Otherwise, it will opt out from the whole thing.

(d) If successes of different types of prototypes are independent, the problem simplifies somewhat. Failing the first prototype doesn't provide any information about the second and we can simply multiply the probabilities. Our decision tree will be identical expect the probabilities in the rightmost branches. This difference stems from the fact that failed protoype on the first try doesn't provide any information about success of the next one.

Starting with a high quality prototype yields expected profits (before sunk costs thus far) of $0.6(-35000+112500)+(1-0.6)\times 0.2(-(15000+35000)+112500)+(1-0.2)(1-0.6)(-15000 - 35000) = 35500$.

Analogously, starting with low quality prototype yields $0.2(-15000 + 112500) + (1 - 0.2) \times 0.6(-(15000 + 35000) + 112500) + (1 - 0.2)(1 - 0.6)(-15000 - 35000) = 33500$. Again, firm doesn't have to try out both, and trying out only high (low) quality option would again yield expected profits of 32500 (7500). Therefore the firm will optimally try high quality first and proceed to low quality one in case of failure. Given this strategy gives and expected profit greater than the cost of developing the blueprints, blueprints will be developed.

Note that the order of trying out the two prototypes is by no means obvious, because there is a trade-off between quality and price. One test has a higher probability of success, and the other test is cheaper. If one alternative were better in both dimensions it would be obviously the first one to try.

13. HJK Helsinki football club sells tickets to its matches on Bolt Arena. Prices range from 5-35 euros depending on the seat, purchase method and discount group, with 30 euros being the de facto price.

Playing a match in front of home crowd has an (fixed) opportunity cost of 10 000 euros: the club could opt for one of the nearby training fields without any stands for audience or other facilities, and save the money paid to Helsinki city for renting the stadium.

The stadium hosts at most 10770 spectators but due to coronavirus only half of the seats can be occupied. Providing the facilities, sanitary, security etc., exhibit only slightly increasing marginal costs. Albeit facility services exhibit also fixed costs, those are miniscule compared to stadium rent or the associated variable costs.

At the moment of selling the tickets sunk costs include all the costs associated with the maintenance, training, and managing the team, federation membership fees, agreements to buy certain services needed for arranging the match (to be fair, most likely stadium rent is one of these unless the club itself can have subtenants) etc.

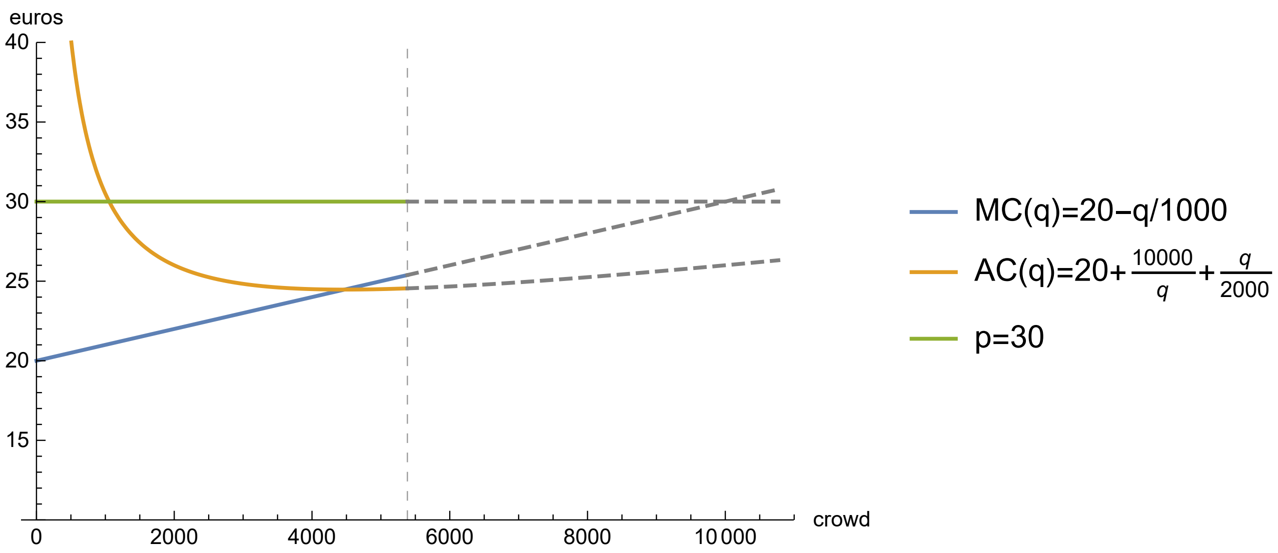


Figure 13: Average and marginal costs as functions of attendance.

14. (a) First notice that the two markets aren't really intertwined. Marginal costs are constant and therefore producing on one market doesn't affect other market's costs. Marketing is done and distribution network built country by country. There's equally strong dichotomy on the demand side as well - selling in one country doesn't satisfy the needs in other.

We have three types of costs here: sunk costs (marketing), fixed costs (having distribution network up) and variable costs (distribution and production cost per unit), which have constant marginal costs. Marketing costs are \$20000 and therefore the recoverable share of entry costs is $600000 - 20000 = 580000$ dollars. The firm faces a fixed opportunity cost of $FC_0 = 600000 \times 0.1 = 60000$ dollars a year for its capital, evaluated pre-entry. Upon entry marketing costs are sunk and fixed cost becomes $FC_1 = 580000 \times 0.1 = 58000$.

Marginal costs are \$10 for unit production plus 2, 12 or 22 euros depending on the market scenario. We assume throughout that both costs and profits are realized in the end of the period. Basically any of the four possible combinations is justifiable as long as these assumptions are articulated.

Let's start with Estonia. We'll first calculate the optimal price and quantity every year given the company has entered. The company will face a marginal cost of $MC_E(q) = MC_E = 10 + 12 = 22$ and collect a marginal revenue of $\frac{\partial}{\partial q}q(125 - 0.05q) = 125 - 0.1q$. Setting these equal and solving for q yields $q_E^* = 1030$. Plugging into demand gives $p_E^* = 73.5$. Yearly profits in the optimum are thus given by $\pi_E^* = (p_E^* - MC_E)q_E^* = 53045$. Present value of the profits before fixed costs, assuming that the profits are realized in the end of the year, $53045(1/(1 + 0.1)) + 53045(1/(1 + 0.1))^2 = 53045/0.1 = 530450$. Market entry costs 600000 and therefore the present value of the entry is -69550 so the company will not enter.[2]

In Latvia, our calculations depend on the assumption, at which time the decision of the volume of the production must be made in the first year. If the firm learns the marginal cost after this decision, the best it can do is to maximize the expected profits $E(\pi(q)) = q(125 - 0.05q) - 10q - (0.5 \times 2q + 0.5 \times 12q) = 103q - 0.05q^2$. FOC becomes $103 - 0.1q = 0 \implies q^* = 1030 \implies p^* = 73.5$. First year profits before fixed costs will be $(73.5 - 12)1030 = 63345$ if cost turns out to be low, $(73.5 - 32)1030 = 42745$ in case of high costs.

Upon learning that the cost is high, it's yearly optimum from second year onwards would be derived as $125 - 0.1q = 32 \implies q^* = 930 \implies p^* = 78.5$. Profits before fixed costs would be $\pi^* = (78.5 - 32) \times 930 = 43245$. If the firm has entered, it can only salvage 580000 of the fixed costs by exiting. Therefore the fixed opportunity cost is $FC_1 = 58000$ dollars a year whereas the firm would only make 43245 a year. Therefore the firm will exit as it can find better use for its money elsewhere.

Present value of firms profits in this scenario is $-20000 - (58000 + 42745)/(1 + 0.1) = -(372550/11) \approx -33868.2$. That is, the marketing cost is sunk, and during the first year an additional 580000 of capital is reserved yielding a fixed opportunity cost of $FC_1 = 58000$. This cost, along with a profit of 42745, are realized in the end of the period. Equivalently, one could think that the firm loses a present value of 600 000 forever, makes little profit in the first year and receives a PV of 580000 in the end of the first year: $-600000 + (580000 + 42745)/(1 + 0.1) \approx -33868.2$.

Should the firm learn that the cost is low, yearly optimum would be $125 - 0.1q = 12 \implies q^* = 1130 \implies p^* = 68.5$. Profits before fixed costs would be $\pi^* = (68.5 - 12) \times 1130 = 63845 > FC_1 = 58000$. Firm would stay in the market, and the present value of the profits would be before entry costs are $63345/(1 + 0.1) + (1/(1 + 0.1))63845/0.1 = 7017950/11 \approx 637995$. Deducting entry costs we get $417950/11 = 37995$.

In expectation the value of entry will be $0.5(-372550+417950)/11 = 188775/11 \approx 2064$. Note that although the costs are similar in expectation in the two countries, the option and exiting in an unfortunate case, i.e. option value of experimentation, makes Latvia

[2] Numbers will change if we assume sales and variable costs are realized in the start of the period. That would move the stream of profits one year back in time, and give entering Estonia a net present value of -16505.

a more lucrative option.

Problem is somewhat simpler if one assumes that the decision on the volume can be made after observing the marginal cost also in the first year. High costs would yield a net present value of $-20000 - (58000 + 42745)/(1 + 0.1) = -(367550/11) \approx -33414$, low costs $-600000 + 63845/0.1 = 38450$. In expectation the value of entry would be $0.5(-367550/11 + 38450) = 27700/11 \approx 2518$.

(b) If the firm were to know the costs, it would only enter when cost is low and could set more favourable prices and quantities in the first period, given it doesn't learn the costs in time. Firm would go on producing that quantity indefinitely. If the cost is high, firm would do nothing at all except possibly paying for that information.

Therefore, paying p_I for the information would be at least good as going blindly to the market if $-p_I + 0.5(-600000 + 63845/0.1) \geq 22700/11 \iff p_I \leq 188775/11 \approx 17161$. If the costs are learnt before the production takes place, $-p_I + 0.5(-600000 + 63845/0.1) \geq 27700/11 \iff p_I \leq 183775/11 \approx 16707$.

15. (a) We need to calculate the present value of a stream of yearly payments of \$1, starting in the first year and lasting for 1 billion years.

A repeating cash flow v that lasts for T years is equivalently (and more conveniently for discounting) interpreted as a perpetual flow that starts this year, minus another perpetuity starting $T + 1$ years from now. Using the present value formula for a perpetuity, this results in

$$\mathrm{PV}(v, r, T) = \frac{v}{r} - \frac{v}{r}\frac{1}{(1 + r)^T} = \frac{v}{r}\Big(1 - (1 + r)^{-T}\Big)$$

where the negative perpetuity was discounted by a further T years. Plugging in the values $v = 1$, $r = 0.03$ and $T = 10^9$ this formula yields a present value of \$33.33.[3]

(b) Let's find the smallest number of years T such that the present value of \$1 per year for T years, discounted at $r = 3\%$, equals 99% of the present value of the billion year stream.

$$\begin{aligned}
\frac{1}{0.03}\Big(1 - (1.03)^{-T}\Big) &> 0.99 \times 33.33 \implies \\
1 - (1.03)^{-T} &> 0.03 \times 0.99 \times 33.33 = 0.99 \implies \\
1 - 0.99 &> 1.03^{-T} \iff \\
\log(0.01) &> -T \log(1.03) \implies \\
\frac{\log(0.01)}{\log(1.03)} &> -T \implies \\
\frac{-4.605}{0.0296} &> -T \implies \\
155.8... &< T
\end{aligned}$$

Therefore the smallest integer number of years needed is 156.

[3] In fact, the present value would be the same to the nearest cent even if the flow only lasted for 300 years.

16. (a) Here we associate the possible outcomes and the respective probabilities in a formal manner. The contestant wins €10 with probability p, $10 \times 2 = 20$ with probability p^2, $10 \times 2^2 = 40$ with probability p^3, etc. A mathematical description of the "lottery" faced by a contestant who plans to stop after S rounds is

$$L = (\{10, 20, 40, \ldots, 10 \times 2^{S-1}\}, \{p, p^2, p^3, \ldots, p^S\}) = \{10 \times 2^{s-1}, p^s\}_{s=1}^{S},$$

where $S \in \{1, \ldots, 12\}$.

(b) If Ukko plans to quit after S questions then his expected utility is
$EU_{S=s} = p^s u(100000 + 10 \times 2^{s-1}) + (1 - p^s)u(100000)$,
where the Bernoulli utility function gives expected utility by weighting the utilities in each state (in this case failure and success) with the respective probabilities. With $S = 2$,
$EU_{S=2} = (1/2)^2\sqrt{100000 + 20} + (1 - (1/2)^2)\sqrt{100000} \approx 316.236$.

Certainty equivalent is the reservation value in terms of a certain amount of money the decision maker would trade the risky lottery for:
$EU_{S=2} = u(100000 + CE_{S=2}) \implies 316.236 = \sqrt{100000 + CE_{S=2}} \implies$
$CE_{S=2} = 316.236^2 - 100000 \approx 5.00$.

If Ukko is never going to quit voluntarily then $S = 12$ and his expected utility is
$EU_{S=12} = (1/2)^{12}\sqrt{100000 + 10 \times 2^{11}} + (1 - (1/2)^{12})\sqrt{100000} \approx 316.235$ and
$EU_{S=12} = \sqrt{100000 + CE_{S=12}} \implies CE_{S=2} = EU_{S=2}^2 - 100000 \approx 4.77$.

Given his risk preferences, Ukko would expect to be better off with a strategy of quitting after two correctly answered questions then with a never-quit strategy.

(c) Ukko is risk averse and will prefer the less risky of two gambles with the same expected value. On any round the expected value of the lottery stays the same as the probability of winning halves while the prize doubles. This gives a hint that the earlier Ukko stops the better.

When considering continuing to round $S > 1$, Ukko is deciding between keeping $u_{S-1} = \sqrt{w_{S-1}} = \sqrt{100000 + 10 \times 2^{S-2}}$ or taking a gamble with expected utility of $E(u_S) = (1/2)(\sqrt{w_S} + \sqrt{100000}) = (1/2)(\sqrt{100000 + 10 \times 2^{S-1}} + \sqrt{100000})$. Ukko's wealth if stopping is exactly halfway between his possible wealths is he continues. Because of concavity of the utility function, it increases faster at lower values of wealth (this is the definition of concavity).

In the first round Ukko has nothing to lose, so he will always take the first question. After that, no matter how many rounds Ukko would be able to play, he would always prefer to stop earlier and will therefore stop exactly after the first question.

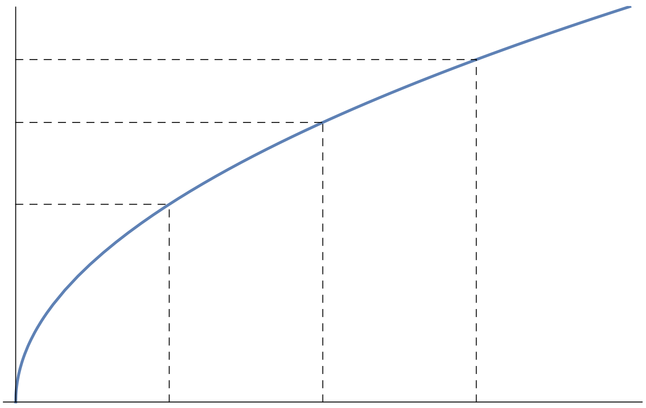


Figure 14: An illustration of a concave function.

You could also use a brute force approach, that is calculate the certainty equivalent (or expected utility) for every $S = 1, \dots, 12$ using the approach seen for cases $S = 2$ and $S = 12$above. That would be a lot of calculations by hand, but if you plug the equation for $EU_{S=s}$ into any numerical program (even Excel will do) it is easy to compare the results for all values of S. It is straightforward to confirm that Ukko gets the highest expected utility from a strategy if quitting after the first question. The certainty equivalent is approximately 5.00 euros, so for practical purposes he is almost indifferent between $S = 1$ and $S = 2$. This is because the 50-50 "gamble" between 0 and 20 euros, while clearly unattractive, is very time compared to his baseline wealth so the risk premium is also subsequently tiny.

(d) Since Akka has a much better than random chance of answering questions correctly, the gamble of continuing is more attractive to her, even though she has the same risk preferences as Ukko. To determine Akka's reservation value, we must first figure out the optimal round for her to stop, S^* and calculate the corresponding expected (Bernoulli) utility. Employing the previous logic, Akka won't stop on the last round given she's got that far. Neither she will stop on the penultimate round et cetera. She'll therefore play all the 12 rounds.

Akka's expected utility from stopping after 12 rounds is given by
$EU_{12} = (3/4)^S \sqrt{100000 + 10 \times 2^{12-1}} + (1 - (3/4)^{12})\sqrt{100000} \approx 317.206.$

We want to know, what is the maximum amount of money Akka would be willing to lose for certain if she gets to play the gamble. Denote the sum by x and the problem can be stated as $\sqrt{100000} = (3/4)^{12}\sqrt{100000 - x + 10 \times 2^{11}} + (1 - (3/4)^{12})\sqrt{100000 - x} \implies x \approx 619.33.$

As in the case for Ukko, you could also use the brute force approach to show this. (Also in that case a complete answer requires showing what formula you used to do the calculations.)

17. (a) The consumer can spend at most $M = 100$,the price of apples $p_a = 0.5$ and $p_b = 1$.

Then it must be that $0.5a + b = 100 \implies b(a) = 100 - 0.5a$. Plugging this into the utility function we get $u(a, b(a)) = a^{\frac{1}{4}}(100 - 0.5a)^{\frac{3}{4}}$.

Keeping eye on (b), we derive the optimality condition for any budget M and price p first. Take the first order condition using product rule and chain rule.

$$\begin{aligned} u'(a, b(a)) = 0 &\iff \\ \frac{1}{4}a^{-\frac{3}{4}}(M - pa)^{\frac{3}{4}} + a^{\frac{1}{4}}\frac{3}{4}(M - pa)^{-\frac{1}{4}}(-p) = 0 &\iff \\ \frac{1}{4}(M - pa)^{\frac{3}{4}}(M - pa)^{\frac{1}{4}} = \frac{3p}{4}a^{\frac{3}{4}}a^{\frac{1}{4}} &\iff \\ M - pa = 3pa &\implies \\ a = \frac{M}{4p} & \end{aligned}$$

Plugging in $M = 100$ and $p = 0.5$ we get $a^* = 50$.

(b) Derived in (a). Demand for apples is $a^d(p, M) = M/(4p)$.

(c) With total expenditure of M the consumer spends $pa^d(p, M) = M/4$ on apples. Therefore the expenditure share of apples is $1/4$.

This is a general property: with a Cobb-Douglas utility function $u(x_1, x_2, \ldots, x_n) = x_1^{\alpha_1} \times x_2^{\alpha_2} \times \cdots \times x_n^{\alpha_n}$, where $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$, the expenditure share on good j is α_j regardless of prices and income.

18. Indifference (aka iso-utility) curve depicts the combinations of goods that yield a given amount of utility. To derive an indifference curve one must first fix the level of utility, $\bar{u}$. Once this is done, we find all the combinations of the goods that yield this exact level of utility.

To make a point that complementarity or substitutability are not properties of the goods but preferences, the same pair of goods with different consumers are used in the following examples

(a) Imperfect substitutes and imperfect complements have different framing or emphasis, but can mean the same thing. You might want to check first the definitions of their perfect counterparts in (c) and (d). Now everything just isn't so black and white: with imperfect substitutes, the level of consumption somewhat determines, how much of one good the consumer is willing to trade for one unit for another. With imperfect complements it's the same story put differently: the goods are best consumed together, but both goods have their place by themselves as well. In real world, basically any pair of goods exhibits these effects.

Angela likes beer and spend time on the terrace, but drinking beer at home or spending time in the terrace without a drink gets boring at some point.

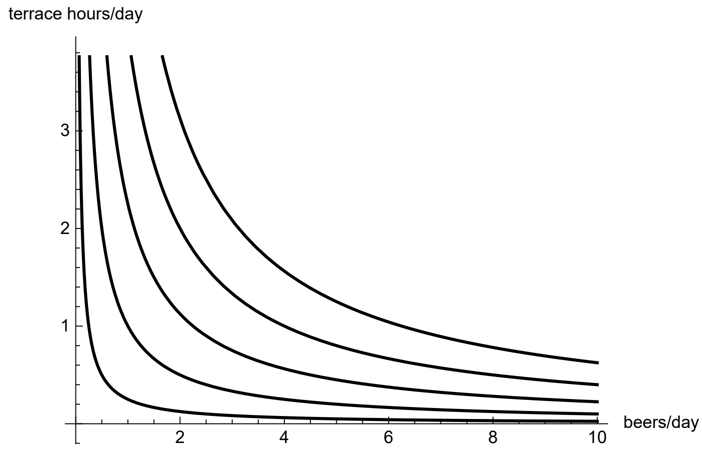


Figure 15: Indifference map of imperfect complements/substitutes.

(b) Two goods are perfect complements if they're always consumed together. One doesn't derive any utility from consuming either of the goods alone. Usually this kind of goods constitute a functional entity, such as left and right shoe, front and rear bicycle wheel etc. Naturally, this, along with perfect substitutes, is an idealized concept and with a little imagination it's easy to come up with use cases for any pair of supposed perfect complements.

Boris finds that beer is of no use drunk at home. On the other hand, terrace without beer is as good as no terrace. For every hour on the terrace Claude wishes to have two beers.

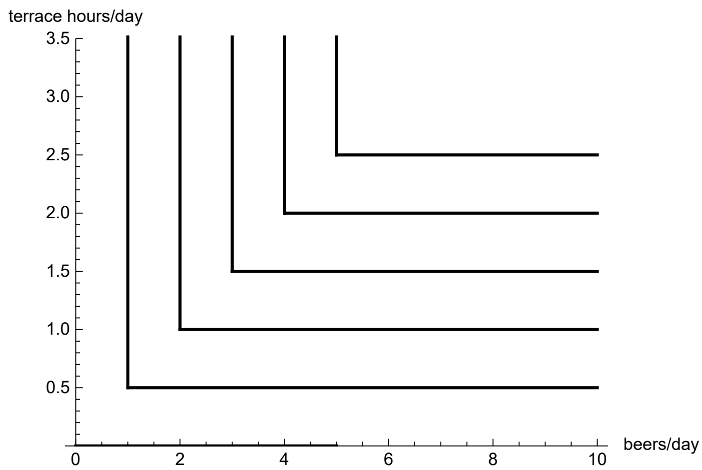


Figure 16: Indifference map of perfect complements.

(c) Two goods are perfect substitutes if the consumer is willing to substitute one good for the other in a fixed ratio, independent of the level of consumption.

Claude will trade a hour of terrace time for 3 beers, or the whole 12-hour daytime for a 36 beers any day, or vice versa. He has no problem drinking his beer on a terrace either but it's just the same for him whether or not he gets to drink on a terrace; terrace is nice as is, as is a beer.

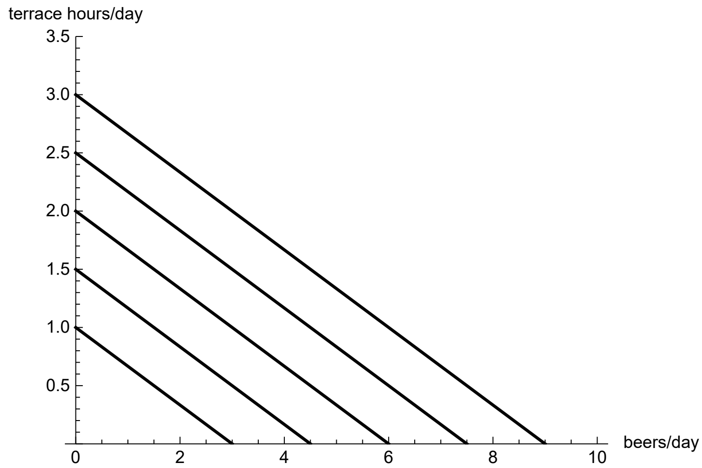


Figure 17: Indifference map of perfect substitutes

19. (a) The firm will produce a quantity at which the marginal cost equals marginal revenue. Because the firm is a price taker (i.e. its output doesn't have effect on the price), marginal revenue simply is the market price of the good, p.

    Optimality condition can be written as $2 + 0.2q = p \implies q^s(p) = 5p - 10 = 5(p - 2)$. The firm will not operate at a loss, so we must have that $\pi(p) \geq 0 \implies p \geq 4$, where we used the expression for profits from (b). Otherwise firm will produce zero output.

    (b) To derive the profits we must figure out total costs. Marginal cost is the derivative of variable costs, so variable costs are the integral of the variable costs.

$$\text{VC}(q) = \int_0^q (2 + 0.2x)\text{d}x = 2q + 0.1q^2$$

To get total costs just add the fixed cost 10. Profits as a function of output price are the difference between revenue and total cost, with the optimal quantity $q^s(p)$ supplied:

$$\begin{aligned}\pi(p) &= \underbrace{pq^s(p)}_{\text{Revenue}} - \big(\underbrace{2q^s(p) + 0.1q^s(p)^2}_{\text{Variable cost}}\big) - \underbrace{10}_{\text{FC}} \\ &= (p-2)q^s(p) - 0.1q^s(p)^2 - 10 \\ &= 5(p-2)^2 - 0.1(25p^2 - 100p + 100) - 10 \\ &= 2.5p^2 - 10p\end{aligned}$$

when $p \geq 4$, and zero otherwise. Notice that the profit function is continuous, even though supply jumps at the break-even price $p = 4$. This is because the firm must be able to cover its fixed costs to supply any stuff at all. The quantity supplied at the break-even price, $q^s(4) = 10$, is known as the minimum efficient scale of production.

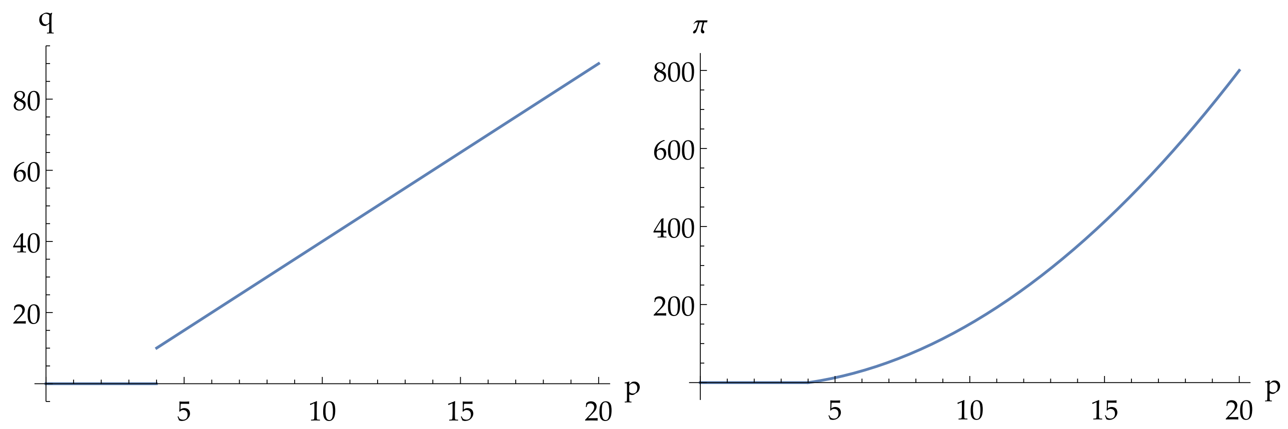


Figure 18: Supply and profit as functions of output price, from parts 19a and 19b respectively.

(c) As more and more firms enter, the market supply curve will shift upwards: at any given price there will be more supply on the market. With demand decreasing in p we know that such shift will lower the market price. As long as price is high enough for firms to earn positive profits, more firms will enter.

In (a) we already derived the price at which firm makes zero profits. Plugging that into the supply curve yields $q^s(4) = 5 \times 4 - 10 = 10$. This is the quantity each firm will produce in equilibrium. Given the price, this is the optimal level of supply as per (a). Therefore best the firm can do is to get zero profits and no firm will benefit from increasing their quantity supplied, nor will any new firm benefit from entering, nor can any existing firm benefit from exiting.

20. (a) Due to nonlinear pricing by the scooter provider, the cost curve will be quite kinky. The consumer has multiple options: take an infinite plan, purchase only hourly or half-hourly plans, go for the minute-based pricing, or have a combination of these.

Let's first consider the costs of the first 60 scooting minutes. Up until $t = 24$ minutes the cheapest method is to go for a minute plan with total cost of $c(t) \leq 0.25 \times 24 = 6$. This equals the cost of one 30 minute package, which the consumer will purchase if $24 < t \leq 30$. Total cost is constant at 6 in this interval.

Stacking new minutes on top of 30 minute plan pays off until 42 minutes of scooting, which costs $30 + 0.25 \times 12 = 9$, which is the cost of 60-minute plan. Therefore the consumer will purchase 60-minute plan if $42 < t \leq 60$ with constant total cost of 9 euros in this interval.

The same pattern goes on until $t = 320$, which has a total cost of $9 \times 5 + 0.25 \times 20 = 50$, which is the cost of unlimited plan, which the consumer will therefore purchase if $t > 320$.

If you wish to do this computationally, the following method may prove useful. Below are listed six plausible ways of combining plans. Plans could be combined arbitrarily, but here we follow the logic above.

With infinite plan total cost is fixed at 50 euros (i), with minute plan total cost will be $0.25t$ (ii), where t is minutes of scooter usage. With only hourly (half hourly) plan [4] $ceil(t/60) \times 9$ (iii) ($ceil(t/30) \times 6$)(iv). This would make $2 \times 9 = 18$ ($3 \times 6 = 18$) euros with $t = 88$, for example.

Combining only hourly and half-hourly plans would yield a total cost of $floor(t/60) \times 9 + 6$ if $mod(t, 60) < 30$, else $ceil(t/60) \times 9$ (v). With $t = 88$, this would make $1 \times 9 + 1 \times 6 = 15$ euros in total. In Figure 19, we'd be on the second short plateau.

Combining minute, half-hourly and hourly plans will yield a cost of $floor(t/60) \times 9 + floor((t - floor(t/60))/30) \times 6 + 0.25 \times mod(t/30)$ (vi). With $t = 88$ this would make $1 \times 9 + 0 \times 6 + 0.25 \times 28 = 16$.

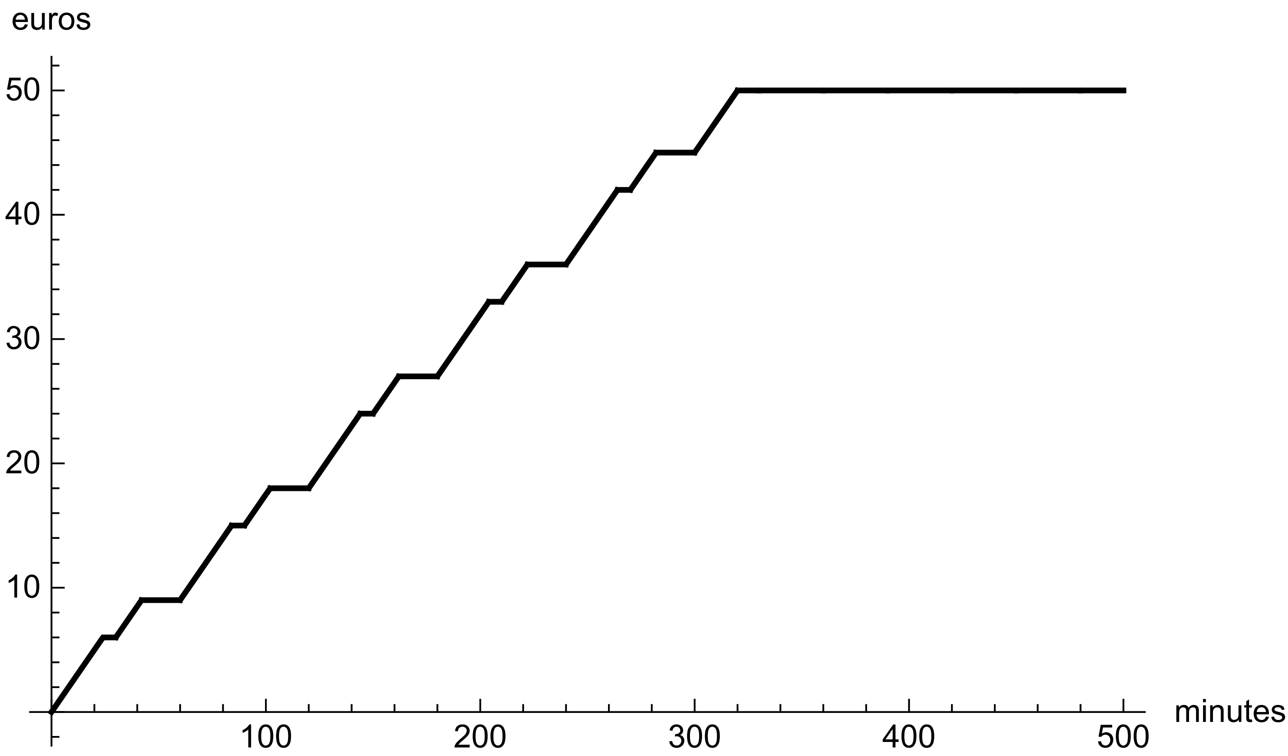


Figure 19: Costs of scooter rental.

(b) Due to the kinky nature of the costs curve also the budget set will exhibit such irregularities. From (a) we know the cost of scooter minutes, $c(t_s)$. Budget constraint stands as $M - p_t t_t - c(t_s) = 100 - t_t - c(t_s) = 0$. Figure 20 depicts the combinations fulfilling this condition. Note that the consumer can have any level of scooting if she decides to have at most 50 minutes of taxi time.

[4] ceil (floor) stands for ceiling (floor) function, which gives the nearest integer greater (smaller) than the argument. For example, ceil(2.2)=3, floor(2.2)=2. Mod gives the remainder of a division, for example mod(123, 60)=3.

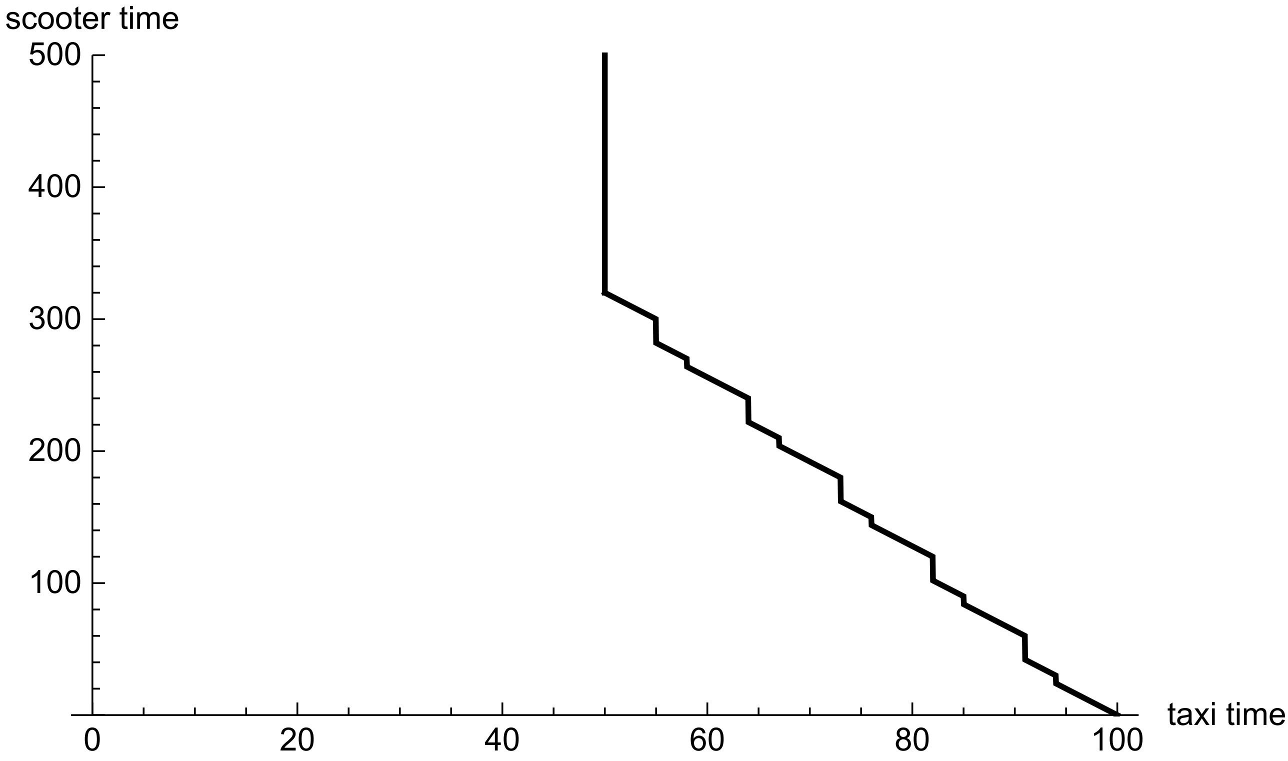


Figure 20: Consumer's budget set.

(c) In the vertical portions of the budget line one doesn't have to give up any taxi time in order to get more scooter time. Looking back at the cost curve, these are the levels of scooter time where one could have more with equal cost. A rational consumer who has smooth preferences (this effectively rules out only perfect complementarity (kink in the indifference curve) and perfect substitution (linear, no curvature) between scooters and taxis) will always strictly prefers the peaks of the kinks in the budget sets to the bottoms.

The more prominent peaks are at even hours, less prominent ones at even half hours. We should expect levels at or just below even hours to be most common with less pronounced mass points at even half hours. You can easily experiment with this by trying to draw by hand a smooth, convex curve that doesn't intersect the budget set and touches it somewhere else than at the aforementioned peaks.

If you do that, you might find that these curves often tangent to the vertical portion of the budget line at $t_t = 50$. These consumers will purchase the unlimited plan.

21. (a) Denote the share of hours allocated to health care by h. Consequently, $1 - h$ can be allocated to other goods implying that $y(h) = 1 - h \implies h = 1 - y$. Plugging this into the production of health care we get the level of life expectancy $x = 20 + 100\sqrt{1-y}$ as a function of other goods produced. (You can equivalently do this the other way around, this just flips the axes in the figure.) This curve depicts all the combinations that use the whole budget. Any point below this curve would be inefficient, as it would be

possible to increase the consumption of health care without reducing the consumption of other goods (and vice versa). Notice that $x = 20$ is achieved “for free”.

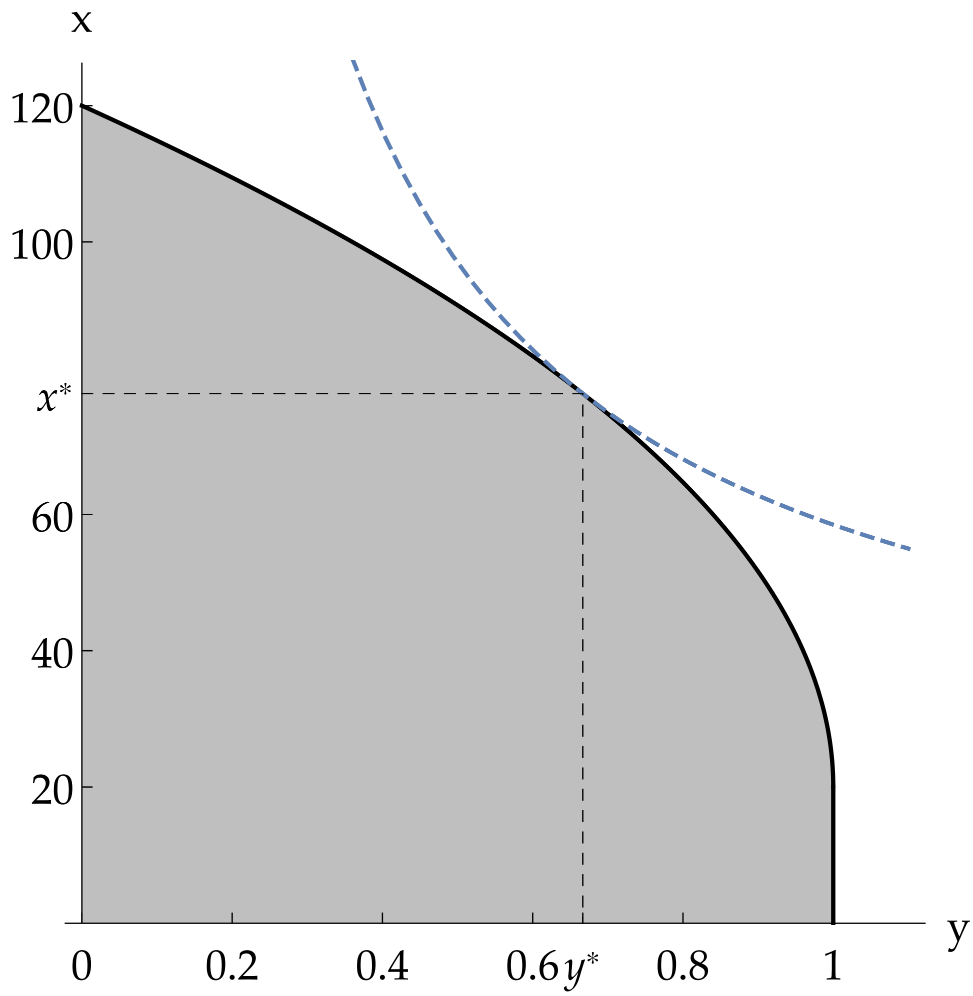


Figure 21: Lilliputians' production possibilities, the highest indifference curve they can reach, and their optimal choice $\{y^*, x^*\}$.

(b) In social optimum Lilliputians' aggregate utility is maximized. As the citizens have a common utility function, this problem coincides with optimizing any individual Lilliputian's utility. We know that the optimum must lie at some point on the production frontier as there's no reward from leaving part of the budget unused.

On this frontier, $x = 20 + 100\sqrt{1-y}$. Plugging this into the utility function yields

$U(y) = (100\sqrt{1-y})^{\frac{1}{2}} y^{\frac{1}{2}}$. The first order condition with respect to y yields[5]

$$\begin{aligned} u'(y) = 0 &\iff \\ 10(\frac{1}{2}y^{-\frac{1}{2}}(1-y)^{\frac{1}{4}} - y^{\frac{1}{2}} \times \frac{1}{4}(1-y)^{-\frac{3}{4}}) = 0 &\iff \\ \frac{1}{2}y^{-\frac{1}{2}}(1-y)^{\frac{1}{4}} = y^{\frac{1}{2}} \times \frac{1}{4}(1-y)^{-\frac{3}{4}}) &\overset{\times y^{\frac{1}{2}}}{\iff} \\ \frac{1}{2}(1-y)^{\frac{1}{4}} = \frac{1}{4}y(1-y)^{-\frac{3}{4}} &\overset{\times 4(1-y)^{\frac{3}{4}}}{\iff} \\ 2(1-y) = y &\iff \\ y^* = 2/3 & \end{aligned}$$

As we have linear production function for y, producing $y^* = 2/3$ requires $2/3$ million worker years of labor. Life expectancy at the optimum is the obtained by plugging the remaining hours $h = 1 - y^* = 1/3$ into $X(h)$: $x^* = X(y^*) = 20 + 100\sqrt{1/3} \approx 78$ years.

22. Throughout this exercise, denote all the curves, surpluses etc. in absence of any subsidies with subscript 0. In the case of \$100 producer (consumer) surplus as in (a) ((b)), denote the altered measures with subscript 1 (2).

    In the case of \$200 producer subsidy as in (c) we use subscript 3. When government intervenes by purchasing milk on the market (d) we use subscript 4.

    All the quantities are in kilotons per year. All surplus measures are yearly.

    (a) Let's start with deriving inverse demand and supply curves in the absence of the subsidy or other interventions: $Q^d(p) = 20 - 0.05p \implies p_0^d(q) = 400 - 20q$, $Q^s(p) = 0.2p - 40 \implies p_0^s(q) = 5q + 200$.

    Equilibrium without subsidy would be $p_0^d(q) = p_0^s(q) \iff 400 - 20q = 5q + 200 \implies q_0^* = 8 \implies p_0^* = p_0^d(8) = 240$.

    Subsidy shifts supply curve downwards: $p_1^s(q) = p_0^s(q) - 100 = 5q + 100$. New equilibrium is obtained as $400 - 20q = 5q + 100 \implies q_1^* = 12 \implies p_1^* = p_0^d(12) = 160$.

    In the absence of the subsidy, $CS_0 = (400-240) \times 8/2 = 640$, $PS_0 = (240-200) \times 8/2 = 160$. Welfare is given by $W_0 = CS_0 + PS_0 = 640 + 160 = 800$.

    With producer subsidy, $CS_1 = (400-160) \times 12/2 = 1440$, $PS_1 = (160-100)(12-0)/2 = 360$. Deadweight loss is given by $(12-8)(240-160)/2 + (12-8)(5 \times 12 + 200 - 240)/2 = 200$. Total amount of subsidy paid is $G = 12 \times 100 = 1200$ and therefore $W_1 = CS_1 + PS_1 - G_1 = 600$.

    Therefore the welfare effects of producer subsidy are $\Delta_1 CS = CS_1 - CS_0 = 1440 - 640 = 800$, $\Delta_1 PS = PS_1 - PS_0 = 360 - 160 = 200$ and $\Delta_1 W = \Delta_1 PS + \Delta_1 CS - G_1 = -DWL_1 = -200$.

[5] Probably an easier way to achieve this would be to note that any value that maximizes a positive-valued

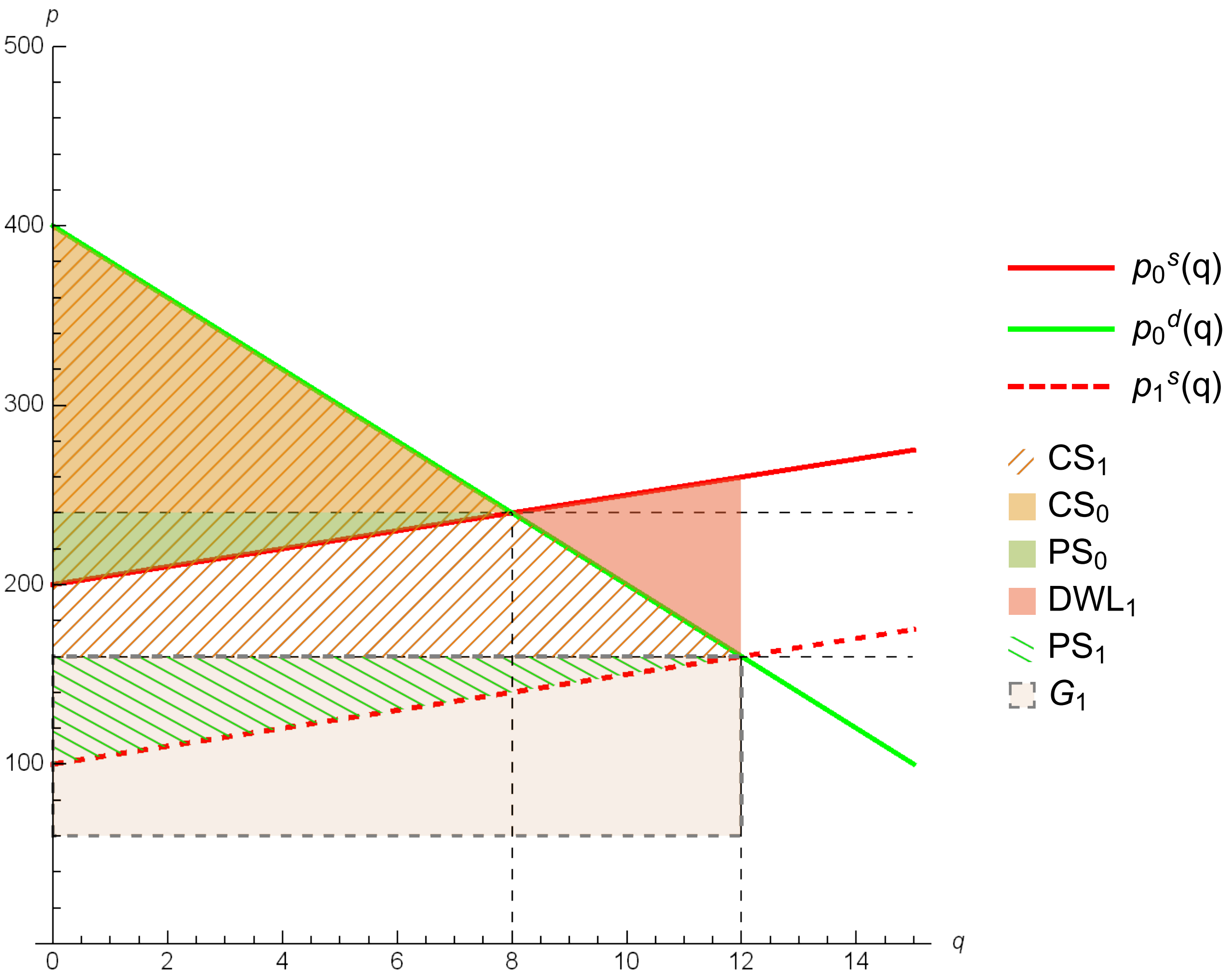


Figure 22: Welfare effects of a \$100/kt unit subsidy, paid to producers.

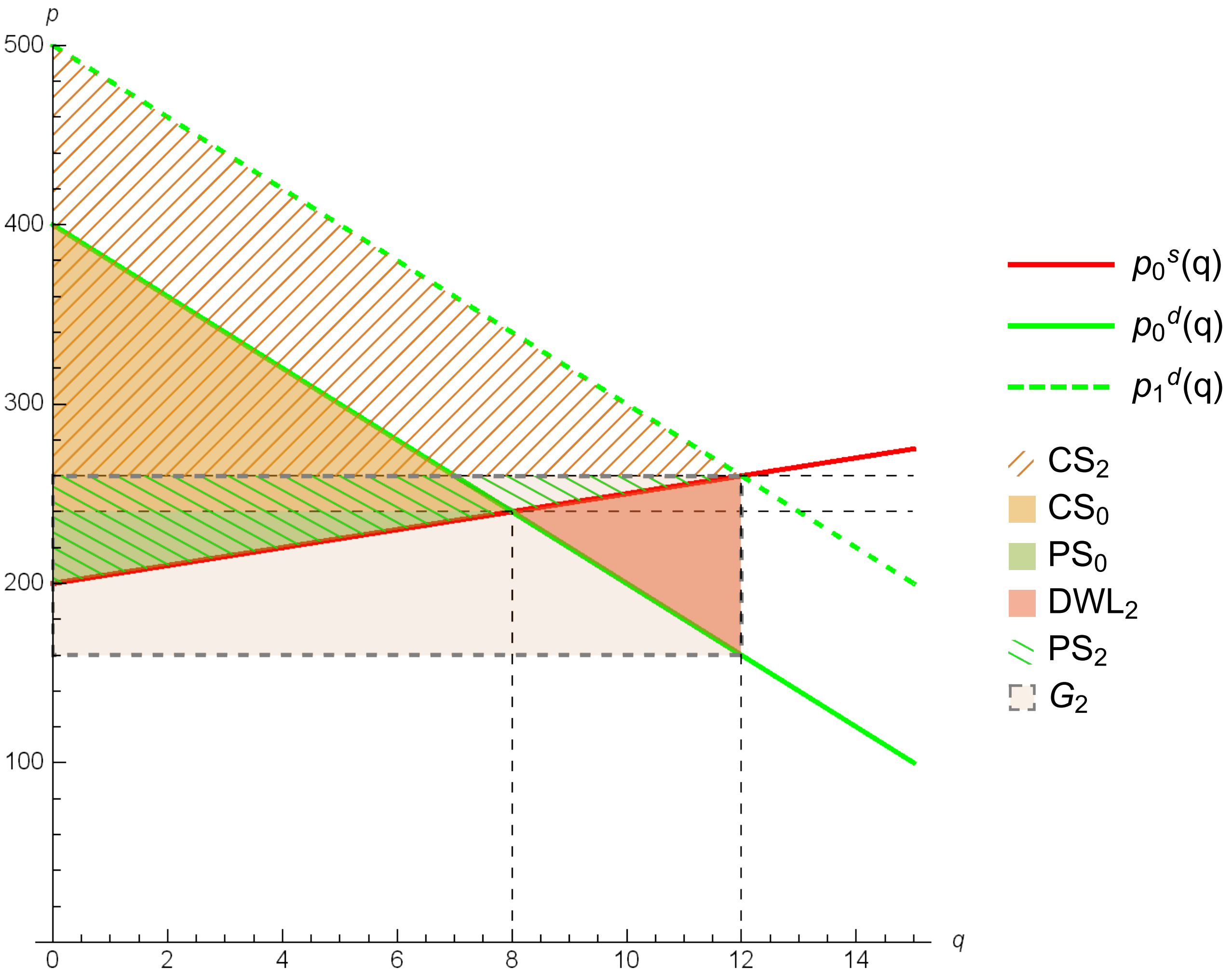


Figure 23: Welfare effects of a \$100/kt unit subsidy, paid to consumers.

(b) Changing the nominal incidence of a subsidy does not change its welfare effects. If you go through the trouble of recalculating everything (not necessary) you find that all areas that capture the monetary values of the components of welfare have the same shape and the same area as when the subsidy was paid to producers.

(c) During the Urban party's reign there are no policy interventions in the market so their welfare effects are zero.

During Farmers' party reign, subsidy shifts supply curve downwards: $p_1^s(q) = p_0^s(q) - 200 = 5q$. The setting is very similar to Figure 22: supply curve only has now shifted another 100 units downwards. New equilibrium is obtained as $400 - 20q = 5q \implies q_1^* = 16 \implies p_1^* = p_0^d(16) = 400 - 20 \times 16 = 80$.

For consumer and producer surplus we obtain $CS_3 = (400 - 80) \times 16/2 = 2560$ and $PS_3 = (80 - 0)(16 - 0)/2 = 640$. Total amount of subsidy paid is $G_3 = 16 \times 200 = 3200$

function also maximizes its fourth power and that a constant multiplier ($\sqrt{100}$) doesn't affect the optimum anyhow. Our problem would stand as $\max_y y^2(1-y)$ which is simpler to analyze.

and therefore $W_3 = CS_1 + PS_1 - G_3 = 0$.

In half of the years there's no welfare loss, in the other half the loss is $\Delta_3 W = W_3 - W - 0 = -800$, yielding an average yearly loss of $(800 + 0)/2 = 400$.

Notice that the average welfare loss from a subsidy that varies across years is much higher than was the welfare loss from a subsidy that is stable at the level of the average subsidy across years 100. This is a general result: the marginal welfare loss from a subsidy (or a tax) increases as the level of the subsidy (or a tax) gets higher.

(d) The government chooses a point on the supply curve where the desired level of surplus is obtained and sizes its purchases accordingly. The only point on the supply curve that yields exactly the same surplus as in (a) is the point we obtained in (a), which is easily verified:

$$\begin{aligned} PS_4 = PS_1 = 360 &\iff \\ (200 + 5q - 200)(q - 0)/2 = 360 &\iff \\ (5/2)q^2 = 360 &\overset{q>0}{\implies} \\ q_4^* = 12 &\implies \\ p_4^* = p_0^s(12) = 200 + 5 \times 12 = 260 & \end{aligned}$$

At this price, quantity sold to consumers is given by $Q^d(260) = 20 - 0.05 \times 260 = 7$. Government will purchase the remaining 5 kilotons.

Counting in the exports, government spending is given by $G = (260 - 40) \times 5 = 1100$. Consumer surplus is given by $CS_4 = (400 - 260)(7 - 0)/2 = 490$ and change in consumer surplus by $\Delta_4 CS = 490 - 640 = -150$. Change in producer surplus is $\Delta_4 PS = 360 - 160 = 200$. In total, the effect of the purchases on welfare is $\Delta_4 W = W_4 - W_0 = 490 + 360 - 1100 - 800 = -1050$.

In general, giving the producers a monetary transfer is a cheaper way to increase producer welfare than incentivizing them to produce costly output that is sold at a loss. Now in addition to paying the producers the government also pays for inefficient excess production.

23. (a) A player picking at speed x l/h obtains €$16x$ of value per hour while suffering effort cost €x^2. While there are chanterelles left the per-hour value function is $16x_i - x_i^2$ where $i \in \{A, B\}$ denotes the player (*A*lice or *B*ernard). To get their payoffs we also need to figure out how many hours they picking will last before the patch is empty. It takes t hours to pick all 24 liters, so that $x_A t + x_B t = 24 \implies t = 24/(x_A + x_B)$. Combining costs and benefits gives the value function

$$V_i(x_i, x_j) = \frac{24}{x_i + x_j}(16x_i - x_i^2) = \frac{384x_i - 24x_i^2}{x_i + x_j},$$

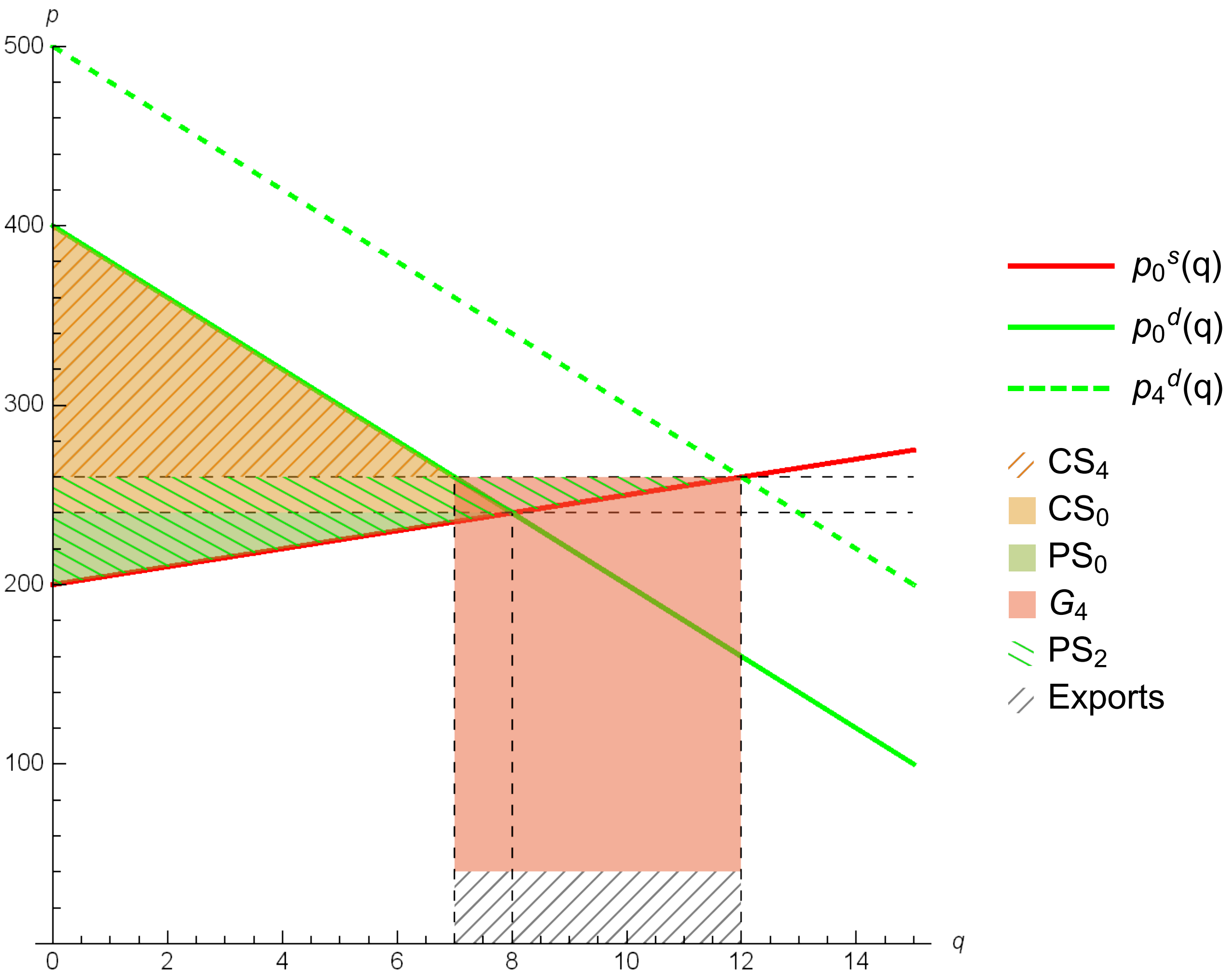


Figure 24: Welfare effects of a price support.

where x_j denotes the choice of the other player. Plugging in the 4×4 combinations of possible actions $x \in \{0, 2, 4, 6\}$ into the value function yields the payoff matrix:[6]

		B: 0	B: 2	B: 4	B: 6
A	0	0, 0	0, 336	0, 288	0, 240
A	2	336, 0	168, 168	112, 192	84, 180
A	4	288, 0	192, 112	144, 144	115.2, 144
A	6	240, 0	180, 84	144, 115.2	120, 120

In Nash equilibrium a unilateral change in strategy is not profitable for any player. First let's see if there are any dominated strategies, i.e., strategies that are never the best response for a player no matter what the other player does. It is quickly apparent that not doing any picking ($x_i = 0$) is not the best response to anything and can be eliminated. This simplifies the payoff matrix to:

		B: 2	B: 4	B: 6
A	2	168, 168	112, 192	84, 180
A	4	192, 112	144, 144	115.2, 144
A	6	180, 84	144, 115.2	120, 120

Now in the top row we see that no matter what Bernard does, 2 is never Alice's best response. Since the game is symmetric, the same holds for Bernard, and the game simplifies further to:

		B: 4	B: 6
A	4	144, 144	115.2, 144
A	6	144, 115.2	120, 120

In the remaining 2×2 game we can quickly find the Nash equilibria by "brute force" reasoning. Considering each of the four possible outcomes in turn, we see that neither would want to deviate from the top left {4,4} or bottom right {6,6} outcomes, so these are Nash equilibria. (There is also a third Nash equilibria, which would involve mixing between speeds 4 and 6, but it is not particularly interesting here so let's ignore it.)

A socially efficient state maximizes the sum of the players' payoffs. While {4,4} is the Nash equilibrium with the highest total payoff, it is not socially efficient. In the full 4×4 payoff matrix {2,2} yields both players a higher payoff, summing to a total payoff of 336. Outcomes where one player picks 0 and the other 2 yield the same total payoff, just all going to one player. Intuitively, the effort cost of picking the chanterelles is minimized at picking speed 2, while benefits are not affected by the (nonzero) speed. With this

[6] It makes sense to do this many formulaic calculations with a computer, e.g., with Excel or Python.

effort cost structure, it doesn't matter how the picking hours are distributed between the players, so all choice combinations {0,2}, {2,0} and {2,2} are socially efficient.

(b) Now that players have what are known as "social" or "other-regarding" preferences, using the symmetric value function derived in part 23a, the value function becomes

$$\hat{V}_i(x_i, x_j) = 0.75V_i(x_i, x_j) + 0.25V_j(x_j, x_i) = \cdots = \frac{288x_i - 18x_i^2 + 6(16 - x_j)x_i}{x_i + x_j},$$

Again, plugging in all 4×4 combinations of choices, the payoff matrix becomes:

A \ B	0	2	4	6
0	0, 0	84, 252	72, 216	60, 180
2	252, 84	168, 168	132, 172	108, 156
4	216, 72	172, 132	144, 144	122.4, 136.8
6	60, 180	156, 108	136.8, 122.4	120, 120

The choice $x_i = 0$ remains dominated, as does $x_i = 2$. This leaves us with the same 2×2-game of undominated strategies as in 23a. However, {6,6} is no longer a Nash equilibrium, since either player can now profitably deviate from it by choosing 4. Checking the remaining outcomes one by one leaves {4,4} as the unique Nash equilibrium in this game. These social preferences (captured by $\beta = 0.25$) are strong enough to deter the most inefficiently speedy picking of chanterelles at 6 l/h, but not strong enough to induce the socially efficient picking speed 2 l/h.

(c) Notice that only Alice's cost and hence her payoffs change. Her value function is now

$$V_A(x_A, x_B) = \frac{24}{x_A + x_B}(16x_A - 0.5x_A^2) = \frac{384x_A - 12x_A^2}{x_A + x_B},$$

while Bernard's is unchanged from part 23a. The payoff matrix is now

A \ B	0	2	4	6
0	0, 0	0, 336	0, 288	0, 240
2	360, 0	180, 168	120, 192	90, 180
4	336, 0	224, 112	168, 144	134.4, 144
6	312, 0	234, 84	187.2, 115.2	156, 120

As before, the zero speed strategies are dominated. As Bernard's values haven't changed picking at 2 l/h is still dominated for him, but since the game is no longer symmetric, this does not guarantee that 2 would also be dominated for Alice. A quick check reveals that 2 remains dominated for Alice as well, so we are again left with a 2×2-game with actions 4 and 6. Now {4,4} is no longer a Nash equilibrium, since Alice could increase her payoff by switching to 6. From {6,4} Bernard has a profitable deviation, and from

{4,6} Alice could deviate to 6 to get a higher payoff. Thus {6,6} is now the unique Nash equilibrium.

Now that Alice has a lower cost of picking the socially efficient outcome has to involve her doing all the picking. She achieves the lowest picking cost at 2 l/h, so {2,0} would the socially efficient choice.

(d) Bernard's ability to publicly commit to a picking a speed turns the situation into a sequential game. In effect, Bernard can pick the column in the full payoff matrix seen in part 23a, knowing that Alice will then pick the row that maximizes her payoff. Here is the payoff matrix with Alice's payoff under her best responses in bold:

		B: 0	B: 2	B: 4	B: 6
	0	0, 0	0, 336	0, 288	0, 240
A	2	**336**, 0	168, 168	112, 192	84, 180
	4	288, 0	**192**, 112	**144**, 144	115.2, 144
	6	240, 0	180, 84	**144**, 115.2	**120**, 120

The only complication for Bernard is that if he were to commit to 4 then Alice is left indifferent between picking 4 or 6. This could lead to Bernard getting a payoff of either 144 or 115.2, one of which is better and the other worse than the guaranteed 120 that he will get from committing to 6. There are two outcomes that can be rationalized as equilibrium outcomes (and either one is acceptable as the correct bottom line answer in this part). In one Bernard commits to 6 and Alice picks 6. In the other Bernard commits to 4 and Alice picks 4. The latter is weak (a bit shaky) in the sense that Alice is has only a weak preference for using the purported equilibrium strategy.

As a side note, in this sequential game Alice's fully formulated strategy consists of a list of responses, one for each of the four possible choices by Bernard. She has two relevant strategies that only differ at her response to $x_B = 4$. If she is playing the strategy where she responds to 4 by 4 then $\{4, 4\}$ is the equilibrium outcome. If she is playing the strategy where she responds to 4 by 6 then $\{6, 6\}$ is the equilibrium outcome. This shows that, as is the case under simultaneous games, there are sequential games where the equilibrium depends on players' the beliefs about what the other player will do.

24. (a) Since the companies are choosing the size of their operations rather than the price, let us first rearrange the demand to get the price at a given supply: $Q = 60 - 12p \implies 12p = 60 - Q \implies p(Q) = 5 - \frac{1}{12}Q$. Note that $Q = q_A + q_B$, i.e. the total amount of unobtainium in the market is the sum of the amounts supplied by the two companies. The profits for Alpha Inc are then given by $\Pi_A(q_A, q_B) = (5 - \frac{1}{12}(q_A + q_B))q_A - 2q_A - 4$ and for Beta Corp symmetrically. Maximizing the profit function with respect to q_A

yields Alpha Inc's best response as a function of Beta Corp's supply:

$$\frac{\partial \Pi_A(q_A, q_B)}{\partial q_A} = \frac{\partial(5q_A - (1/12)q_A^2 - (1/12)q_A q_B - 2q_A - 4)}{\partial q_A} = 0$$
$$\implies 5 - \frac{1}{6}q_A - \frac{1}{12}q_B - 2 = 0$$
$$\implies q_A^*(q_B) = 30 - \frac{1}{2}q_B - 12$$

By symmetry the best response function for Beta Corp is $q_B^*(q_A) = 30 - \frac{1}{2}q_A - 12$. To figure out the equilibrium supplies, we can then plug one company's best response function into the other's:

$$q_A^*(q_B^*(q_A)) = 30 - \frac{1}{2}(30 - \frac{1}{2}q_A - 12) - 12$$
$$= 30 - 15 + 6 - 12 + \frac{1}{4}q_A$$
$$\implies (1 - \frac{1}{4})q_A = 9 \implies q_A = 12$$

Again by symmetry, $q_B = 12$ as well, implying that $Q = 12 + 12 = 24 = 60 - 12p \implies p = 3$. The profits for both companies are thus $\Pi_i = 3 \times 12 - 2 \times 12 - 4 = 8$.

(b) Beta's profit function after the investment would be $\Pi_B(q_B, q_A) = p(q_A, q_B)q_B - 1.5q_B - 6$. Similar derivation as in 24a yields Beta's new best response function $q_B^*(q_A) = 30 - \frac{1}{2}q_A - 9$. If beta hides the investment from Alpha, Alpha will continue to supply $q_A = 12$, making Beta's best response $q_B = 15$, which implies that $Q = 27 = 60 - 12p \implies p = 2.75$. Beta's profit is then $\Pi_B = 2.75 \times 15 - 1.5 \times 15 - 6 = 12.75$, which is more than it made without the investment, so making the investment and hiding it from Alpha is worth it.

However, it remains to check whether Beta would want to hide its investment. If it doesn't, Alpha's best response can be figured out by plugging Beta's (correct) best response functtion to Alpha's best response function from 24a:

$$q_A^*(q_B^*(q_A)) = 30 - \frac{1}{2}(30 - \frac{1}{2}q_A - 9) - 12$$
$$= 30 - 15 + 4.5 - 12 + \frac{1}{4}q_A$$
$$\implies (1 - \frac{1}{4})q_A = 7.5 \implies q_A = 10$$

This means that Beta's best response is $q_B = 30 - 5 - 9 = 16$, and $Q = 26 = 60 - 12p \implies p = \frac{17}{6}$, so it makes a profit of $\Pi_B = \frac{17}{6} \times 16 - 1.5 \times 16 - 6 \approx 15.33$. Thus, Beta makes an even higher profit when Alpha knows about its investment.

(c) As Alpha gets to launch it's ship first, Beta can only react to whatever Alpha did with its best response. Knowing this, Alpha can simply plug Beta's best response straight into it's profit function:

$$\Pi_A(q_A) = 5q_A - (1/12)q_A^2 - (1/12)q_A(30 - \frac{1}{2}q_A - 12) - 2q_A - 4$$
$$= -\frac{1}{24}q_A^2 + \frac{3}{2}q_A - 4$$

Maximizing this with respect to q_A yields Alpha's strategy:

$$\frac{\partial \Pi_A(q_A)}{\partial q_A} = \frac{3}{2} - \frac{1}{12}q_A = 0$$
$$\implies q_A = 18$$

This means that $q_B = 30 - \frac{1}{2} \times 18 - 12 = 9$ and $Q = 18 + 9 = 27 = 60 - 12p \implies p = 2.75$. The profits are then $\Pi_A = 2.75 \times 18 - 2 \times 18 - 4 = 9.5$ and $\Pi_B = 2.75 \times 9 - 2 \times 9 - 4 = 2.75$.

(d) Suppose Alpha launches first. If Beta's investment is hidden, Alpha still thinks Beta is going to respond as if it had not made the investment. Hence Alpha's strategy is exactly the same as in 24c, i.e. $q_A = 18$. Meanwhile Beta's best response function is now as in 24b: $q_B^*(q_A) = 30 - \frac{1}{2}q_A - 9 \implies q_B^*(18) = 30 - \frac{1}{2} \times 18 - 9 = 12$. This means that $Q = 30 = 60 - 12p \implies p = 2.5$. The equilibrium profits are $\Pi_A = 2.5 \times 18 - 2 \times 18 - 4 = 5$ and $\Pi_B = 2.5 \times 12 - 1.5 \times 12 - 6 = 6$. What if Beta had not hidden the investment? Then Alpha would plug the correct best response for Beta in its profit function and maximize

$$\Pi_A(q_A) = 5q_A - (1/12)q_A^2 - (1/12)q_A(30 - \frac{1}{2}q_A - 9) - 2q_A - 4$$
$$= 1.25q_A - \frac{1}{24}q_A^2 - 4$$

Maximizing this with respect to q_A yields Alpha's strategy:

$$\frac{\partial \Pi_A(q_A)}{\partial q_A} = 1.25 - \frac{1}{12}q_A = 0$$
$$\implies q_A = 15$$

Beta's best response is $q_B^*(18.5) = 30 - \frac{1}{2} \times 15 - 9 = 13.5$, which means that $Q = 28.5 = 60 - 12p \implies p = 2.625$. The equilibrium profits in this case would be $\Pi_A = 2.625 \times 15 - 2 \times 15 - 4 = 5.375$ and $\Pi_B = 2.625 \times 13.5 - 1.5 \times 13.5 - 6 = 9.1875$.

Suppose then Beta launches first. Notice that from the point of view of Alpha's response, it doesn't matter if Beta has made the investment or not, or if Alpha knows about the investment or not. All Alpha cares about is the actual quantity supplied by Beta. It's best response function is exactly the same as in 24a: $q_A^*(q_B) = 30 - \frac{1}{2}q_B - 12$. Beta plugs this into its post-investment profit function $\Pi_B(q_B, q_A) = p(q_A, q_B)q_B - 1.5q_B - 6$ and maximizes with respect to q_B.

$$\frac{\partial(5q_B - (1/12)q_B^2 - (1/12)q_B(30 - \frac{1}{2}q_B - 12) - 1.5q_B - 6)}{\partial q_B} = 2 - \frac{1}{12}q_B = 0$$
$$\implies q_B = 24$$

Alpha's best response is then $q_A^*(24) = 30 - \frac{1}{2} \times 24 - 12 = 6$, which means that $Q = 30 = 60 - 12p \implies p = 2.5$ and the profits are $\Pi_A = 2.5 \times 6 - 2 \times 6 - 4 = -1$ and $\Pi_B = 2.5 \times 24 - 1.5 \times 24 - 6 = 18$. Note that the firms are only deciding on their capacity - the fixed costs of building the ship are sunk - so Alpha will provide $q_A = 6$ even when that means a loss (any other $q_A$would yield an even larger loss).

25. (a) With two players and only one year left, we can represent the game with the payoff matrix:

1 \ 2	Stay	Exit
Stay	-10, -10	40, 0
Exit	0, 40	0, 0

There are two pure strategy Nash equilibria in this game, {Stay, Exit} and {Exit, Stay}, as neither player can profitably deviate from those states unilaterally. There is also a mixed strategy Nash equilibrium, where both firms exit with some probability. Since the game is symmetric, the probability will be the same for both players. The probability p in the equilibrium must be such that the firms are indifferent between their pure strategies, i.e. their expected payoff from staying and exiting is the same when the other player exits with probability p. The payoff from staying is $40p + (1-p)(-10)$ whereas the payoff from exiting is zero. The equilibrium p can thus be solved from $40p + (1-p)(-10) = 0 \implies p = 1/5$.

(b) In each period, the highest possible profit is made when there is a single firm in the market, and it equals $100/1 - 60 = 40$ million euros. With a discount rate of $r = 0.05$ and an infinite horizon, the industry could have a present value of profits of $40/0.05 = 800$ million euros if one of the firms exited immediately.[7]

(c) In the infinitely repeated version of the game, at the start of each period the highest feasible present value calculated in 25b is the payoff that the firm will receive if they stay in the game and the other firm exits. If both they and the other firm stay, they will incur a loss of $100/2 - 60 = -10$ and move on to the next period. Note that we can start analyzing the game from any period, because any costs accumulated in the past are sunk and hence irrelevant for the players' decisions going forward, and the future always looks the same in terms of payoffs going forward. This game has two pure strategy subgame perfect equilibria: one where one firm stays in every period and the other one exits, and another where the roles are reversed. These equilibria are not symmetric, however, since the players are not using the same strategy. To find a symmetric equilibrium, we need to look for a mixed strategy one.

[7] Here you could just as well assume that the first payoff is not discounted (from the end of the period), giving a present value 840 million, resulting in similar difference for subsequent present values. Here both answers are equally acceptable. Notice that this difference in interpretations cannot make a difference to any choice or equilibrium, because it amounts to multiplying all payoffs by all players by the same positive number.

In a mixed strategy equilibrium a firm exits with some probability p and stays with probability $1-p$. In a symmetric equilibrium, this probability will be the same for both firms. The equilibrium p has to make the other firm indifferent between staying and exiting - otherwise it could make a profitable deviation. When one firm exits with probability p, the other firm's payoff from staying is $800p+(1-p)(-10+V)$ where V is the continuation value, i.e. the expected value the firm will obtain from continuing the game (discounted by one period). Meanwhile, the payoff from exiting is 0. Notice, that since in a symmetric equilibrium both firms will be mixing, they must be indifferent between staying and exiting also in the next period. Because the payoff from exiting in the next period is zero, the expected value of staying must also be zero, otherwise the firm would not be indifferent between them. This means that in the symmetric equilibrium $V=0$, and we can ignore the continuation value in the derivation of the equilibrium. The equilibrium p can then be solved by equalizing the payoffs from staying and exiting:

$$800p+(1-p)(-10)=0 \implies p=\frac{1}{81}\approx 0.012.$$

The symmetric equilibrium is one where both firms' strategy is to exit with this small probability in each period.

As firms are indifferent between staying and exiting in a mixed strategy equilibrium, it is enough to check the expected value under one of them. As the expected value from exiting is zero, so must be the expected value from staying, for both firms and for the industry. Finally, the expected present value of zero forever is clearly zero. In effect, the chance of obtaining monopoly profits in the future is in expectation exactly squandered by the delayed exit from the initially loss-making duopoly.

Side comment: there are also two asymmetric (and efficient) equilibria, where one firm exits immediately and the other stays.

26. Alex has won a raffle for two free movie tickets. He sees this as an opportunity to ask out Betty, whom he would very much like to spend time with. The problem is that there are only two movies playing - "Up" and "Down" - and Alex and Betty have the exact opposite preferences over them. While Alex is a big fan of heartfelt animations, Betty finds them insufferable, and would instead love to see the Dutch-American sci-fi horror film about evil elevators, which to Alex sounds terrible. Both get a payoff of v from seeing their preferred movie and a payoff of $-c$ from seeing the movie the other one prefers. However, they also value the time spent with each other at w. Alex must first choose whether or not he will ask Betty to the movies with him. He can also choose to keep both of the tickets and go see "Up", which he enjoys the same every time he sees it, twice. If he decides to ask Betty to go with him, Betty must choose whether to accept the invitation or not. If she doesn't, she gets a payoff of zero while Alex goes to see "Up" twice. If she does, Alex, as the owner of the tickets, will get to choose which movie they go to see. The game is illustrated by the

tree in Figure 25.

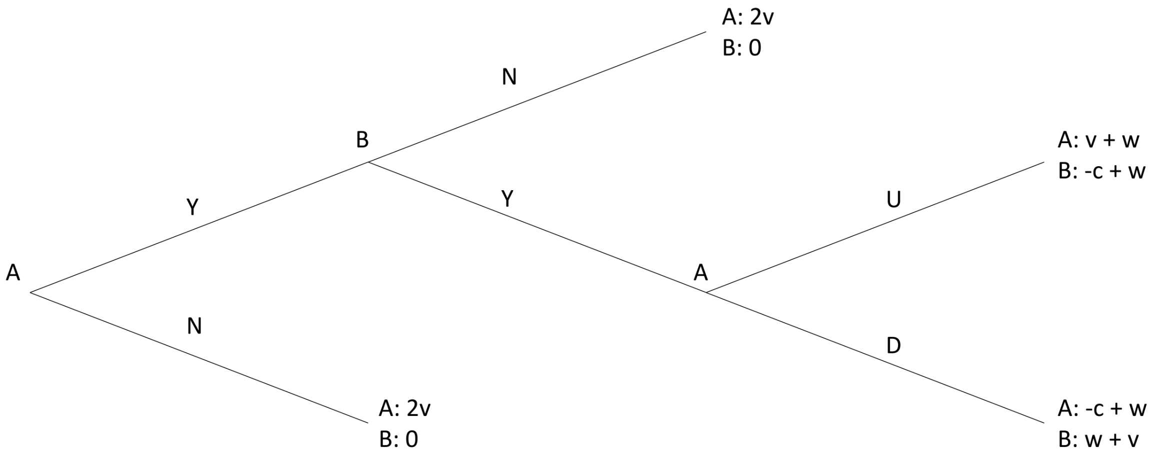


Figure 25: Game tree for Alex and Betty.

Y denotes Alex answering "Yes" to the question of whether or not he will ask Betty to the movies while N denotes the answer "No". Similarly for Betty, N means she will not accept Alex's invitation and Y means that she will. Note that Alex has a total of $2^2 = 4$ strategies: YU, YD, NU and ND. Betty only has two strategies: N and Y.

There are a few things we can say about equilibria in this game without putting explicit values on the payoffs. First, it is clear that "Down" will never be seen in a subgame perfect equilibrium (despite how intriguing the plot synopsis sounds!): assuming $c, v, w > 0$, it must be that $w + v > -c + w$, so the Nash equilibrium in the smallest subgame consisting of only Alex's move will always have Alex choosing "Up". Considering then the second-largest subgame where Betty moves first, we can see that she will choose Y if the value of spending time with Alex exceeds the cost of seeing "Up", i.e. if $w > c$.

Suppose it is the case that $w > c$. Then in the largest subgame, i.e. the full game, Alex will choose Y in his first move if $v + w > 2v \implies w > v$, i.e. the value of spending time with Betty exceeds the value of seeing "Up" again. Hence, depending on the values of c, v and w, the game can have different subgame perfect equilibria:

- If $v > w$ and $w > c$, the subgame perfect equilibrium is {NU,Y}
- If $v > w$ and $w < c$, the subgame perfect equilibrium is {NU,N}
- If $v < w$ and $w > c$, the subgame perfect equilibrium is {YU,Y}
- If $v < w$ and $w < c$, the subgame perfect equilibrium is {YU,N}

What if $w = v$ or $w = c$? Then several of the above can be subgame perfect equilibria (which of them are depends on whether $w = v$ or $w = c$ and what is the relationship of the other parameter-pair. If $w = v = c$, they are all subgame perfect equilibria).

Notice that in addition to the subgame perfect equilibria, there are numerous Nash equilibria that are not subgame perfect. For instance, if $v > w$, then any combination of strategies where Alex picks N in the first round is a Nash equilibrium, because neither player can unilaterally improve on the payoffs $\{2v, 0\}$.

27. (a) Recall the payoff matrix from 6.1a but with the action 6 no longer available.

		B: 0	B: 2	B: 4
A	0	0, 0	0, 336	0, 288
	2	336, 0	168, 168	112, 192
	4	288, 0	192, 112	144, 144

A finitely repeated game can be solved by backwards induction. We know from 6.1a that the one-shot game has a single Nash equilibrium, {4,4} (since {6,6,} is no longer available). That is the subgame perfect Nash equilibrium (SPNE) in the last period, hence also in the second to last period, etc. Thus in the SPNE both players always choose to pick chanterelles at 4 l/h.

Additional intuition. The total possible payoffs for both players in period 5 consist of the payoffs from the above matrix plus some constant, which represents the sum of their payoffs from previous periods. However, the important point is that they can't affect the constant with their choices in the last period, and the constant is the same for all states, so only the above matrix matters, and the same reasoning as in 6.1a applies. There is a single Nash equilibrium, {4,4}, in each of the last-period subgames (note that there are many of these: one per each possible history of choices leading up to that point). Going back to period 4, we have almost the same situation, except now there is also a continuation value (the value the players will gain from period 5) associated with each state the players might end up in. However, we already know that what happens in period 5 in a SPNE does not depend on what the players have done prior to that - the continuation value will be the same regardless of what is done in this period. Hence, the above matrix is again all that matters for the player's choices. This same logic can be repeated going back one period at a time all the way back to the first period - any possibility of cooperation unravels because the players know that they will not cooperate in the last period, so there's no incentive to cooperate in the period before that, so they also shouldn't cooperate in the period before that etc.

(b) Backwards induction is not possible in a game without a known final period. This opens up opportunities for cooperation, as a threat of lost benefits of future cooperation can be used to enforce cooperation in the present. We saw in 6.1a that the best outcome for the players comes from playing {2,2}, which results in a period payoff of $V^* = 168$ for both. This is the cooperative or socially efficient payoff, but is not a Nash equilibrium. The Nash equilibrium {4,4} results in $V^0 = 144$ for both. And a player that "cheats" by deviating from the cooperative outcome obtains $V^c = 192$.

The strongest possible "punishment" is meted out by the Grim trigger strategy, where a player starts by cooperating (here: choose 2), but any deviation by the other player will result in a permanent switch to playing the Nash equilibrium strategy (choose 4). Form {Grim,Grim} to be an equilibrium it must yield a higher present value than cheating followed by Nash equilibrium forever after. If both play Grim then, using the perpetuity formula, and denoting the discount factor $B = 1/(1+r)$, the present value of a player is

$$\Pi^* = 168 + 168B + 168B^2 + \cdots = 168 + \frac{168}{r} = 168 + \frac{168}{0.1} = 1848.$$

A cheater would get V^c in the first period and then V^0 forever after, resulting in present value

$$\Pi^c = 192 + 144B + 144B^2 + \cdots = 192 + \frac{144}{r} = 1632.$$

As cheating against Grim strategy is not attractive if done in the first period, it cannot improve PV later either.

(c) Now Alice and Bernard will get the same payoff as in parts 27a and 27b, but with only 50% probability, and a payoff of 0 with 50% probability. This does not change any comparison between strategies, as all payoffs are effectively cut in half. Equilibrium strategies are not affected.

(d) Now that Bernard is the less patient player, his patience will be the limiting factor on the ability to sustain cooperation. For Bernard to not be too tempted by the immediate payoff from cheating, his present value from cheating must not exceed the PV from permanent cooperation. Let's use the PV formulas from 27b but leave in the discount rate r as an unknown. To ensure that the PV from cheating does not exceed the PV from {Grim,Grim} we need

$$168 + \frac{168}{r} > 192 + \frac{144}{r},$$

from which we can solve r as the unknown: $r < 1$. This means that as long as Bertrand's discount rate does not exceed 100% cooperation can be sustained indefinitely.

28. (a) If both vendors charge the same amount for their ice cream, the consumers will choose the vendor closest to them (or not buy anything at all). Suppose a customer is located at the western end. Their cost of shopping at Abholos is $3 \times 0.5 + 2 = 3.5$, which is less than their reservation value, so Abholos will get all the customers from the western end, giving it a profit of $300 \times (2 - 1) = 300$. A customer at the eastern end, meanwhile, has a cost of shopping at Bokrug of $4 \times 0.5 + 2 = 4$, which is still less than their reservation value, so Bokrug will get all the customers from the eastern end at a profit of $400 \times (2 - 1) = 400$. Clearly, everyone on the 300 meters of beach between Abholos

and Bokrug will also want ice cream (their distance to the closest vendor is shorter than at either border). The customers are split in half, yielding each vendor a further $150 \times (2-1) = 150$ in profit. Abholos' total profit is then $300+150 = 450$ and Bokrug's $400 + 150 = 550$. Consumer surplus is depicted in Figure 26, where the blue line represents the consumer surplus of shopping at Abholos and the red line the consumer surplus of shopping at Bokrug for the customers located on the beach at a point on the horizontal axis. The customer at the western end of the beach, for example, gets consumer surplus of $5 - 3 \times 0.5 - 2 = 1.5$ by shopping at Abholos. At every location customers choose what gives them the highest surplus, so total consumer surplus is represented by the area under the blue curve up to 450 meters (0.45 km) plus the area under the red curve from that point on.

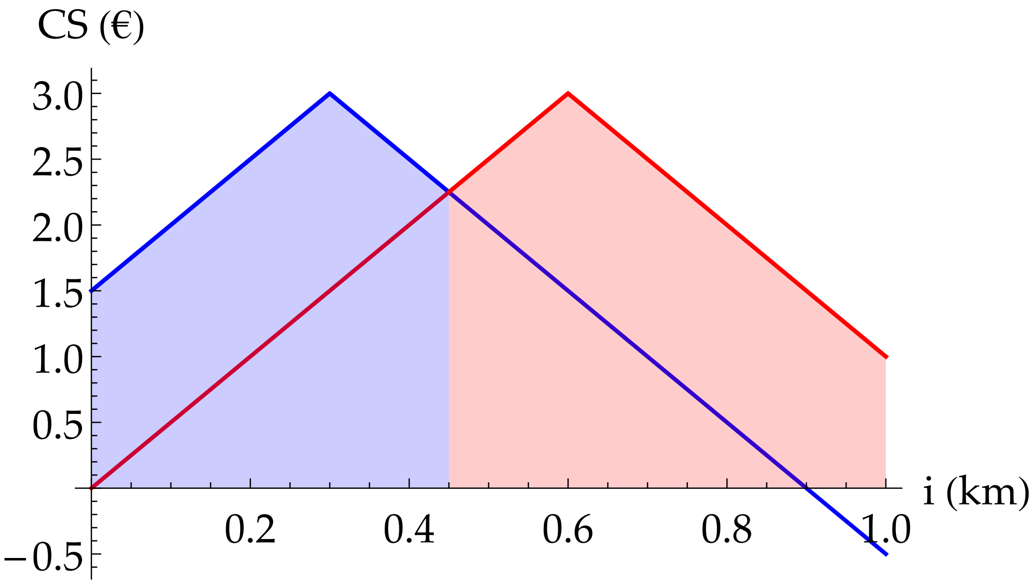


Figure 26: Consumer surplus at Shell Beach. Blue curve depicts the CS of a consumer located at i if they were to buy from Abholos, red curve if they buy from Bokrug, while both are charging €2.

(b) Note that Abholos will not lose any customers by locating somewhere between where it currently is and where Bokrug is: if both vendors were located where Bokrug is, consumer surplus would still be at least zero even at the western border of the beach—since $6 \times 0.5 + 2 = 5$ —and as long as Abholos is to the west of Bokrug, customers on its west side will choose it. (By the definition of a reservation price, a consumer who gets 0 surplus still buys the product.) Meanwhile, any customers between Abholos and Bokrug will always be split evenly between them. Hence, with each meter Abholos moves towards Bokrug, it will gain customers from east and not lose anyone from west. This means that Abholos should locate right next to Bokrug on the western side, yielding it a profit of $599 \times (2-1) = 599$. (Locating on the eastern side of Bokrug would cut its profits to below what Bokrug earns in 28a, because it would lose all west-side customers, so it is not profitable.)

(c) Let p denote Bokrug's price, while Abholos price is fixed at 2. Notice first from Figure 26 that in order to capture customers on the west side of Abholos, Bokrug would need to set its price low enough that consumer surplus at the western border rises above 1.5. This would happen when $5 - (6 \times 0.5 + p) > 1.5 \implies p < 0.5$, which, since the marginal cost of ice cream is 1, is clearly not profitable for Bokrug. Hence, we can rule out Bokrug trying to capture all customers.

Consider then the customers located between Abholos and Bokrug. Let x be the distance in hundreds of meters a customer is located from Bokrug on its western side. The customer will choose Bokrug if $0.5x+p < 0.5(3-x)+2 \implies p < 3.5-x \implies x < 3.5-p$. Hence the length of beach Bokrug controls on its west side at price $p < 3.5$ is $3.5 - p$ hundred meters, and its west-side profit $(p-1)(3.5-p) = 4.5p - p^2 - 3.5$ hundred euros. When $p > 3.5$, Bokrug loses all of its customers to Abholos. At exactly $p = 3.5$ Bokrug and Abholos would split the customers to the east of Bokrug, but this can not be profit-maximizing for Bokrug, since a one cent price reduction would roughly double its sales.

Next, notice from Figure 26 that Bokrug will not lose any customers from its eastern side until consumer surplus on the eastern border of Shell Beach drops below zero, which will happen when $4 \times 0.5 + p = 5 \implies p = 3$. Hence Bokrug's profit from its eastern side is $4(p-1)$ hundred euros when $p \leq 3$.

Finally, let y stand for the distance in hundreds of meters that a customer is located from Bokrug on its eastern side. The customer will still shop at Bokrug if $0.5y + p < 5 \implies p < 5 - 0.5y \implies y < 10 - 2p$. Hence the length of beach Bokrug controls on its east side at price $p \in [3, 3.5]$ is $10 - 2p$ hundred meters and its east-side profit $(p-1)(10-2p) = 12p - 2p^2 - 10$ hundred euros. With these pieces, we can construct Bokrug's profit function (ignoring the obviously unprofitable case of charging below marginal cost, $p < 1$) can be written in hundreds of euros as:

$$\Pi_B(p) = \begin{cases} 4.5p - p^2 - 3.5 + 4(p-1) & \text{if } p \in [1,3) \\ 4.5p - p^2 - 3.5 + 12p - 2p^2 - 10 & \text{if } p \in [3, 3.5) \\ 0 & \text{if } p \geq 3.5 \end{cases}$$

When $p \in (1,3)$ Bokrug's profits are increasing in p, which can be seen from evaluating the derivative of Π_B there: $8.5 - 2p$ is decreasing in p but still positive at $p = 3$.

When $p \in (3, 3.5)$ profits are decreasing in p, as $\Pi_B'(p) = 16.5 - 6p$, which is negative already at $p = 3$ and further decreasing beyond that.

Hence Bokrug's profits are maximized by charging €3 for its ice cream, which results in a profit of $\Pi_B(3) = 9$. Since here we measured distance in hundreds of meters, there are then 100 customers per each unit of distance, this translates into a profit of €900.

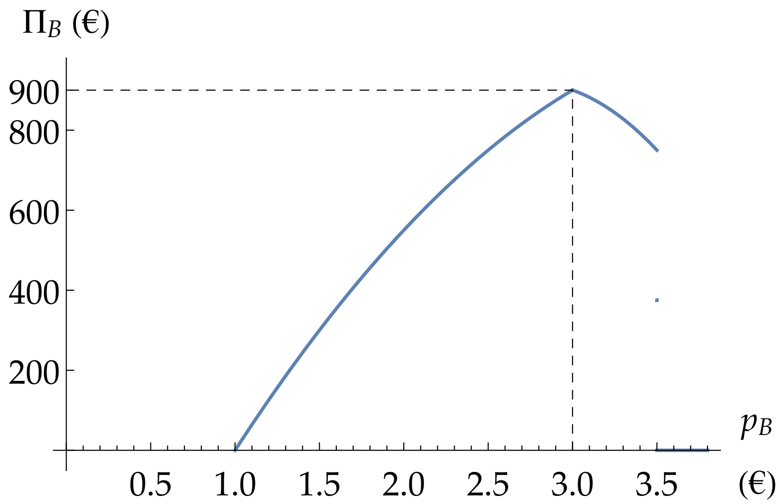


Figure 27: Bokrug's profits as a function of its price, when Abholos has set its price at €2.

29. A classic example of mixed bundling is the McDonald's (or Burger King, or Hesburger, or...) meal. The burger, soda and fries are all available individually, but buying them together is cheaper. As most people probably go to burger joints for the burger, it makes sense to try to sell them fries and soda as well - both are high margin complements for the burger. Meanwhile, if a customer is only looking for a light snack or something to drink, their willingness to pay for the soda and the fries as single items is likely higher than for people who are mainly interested in the burger. There is also not much worry about a secondary market for such perishable (ignoring all the tests where McDonald's meals stay edible for ages in room temperature) items.

    Mixed bundling in this case is usually combined with a quantity discount - the "plus meal". The idea is to sell high-appetite people even more fries and soda - again, very high margin items even with the discounts - with a relatively small increase in price (usually something salient like 1 euro). Beyond likely psychological factors, the price increase is small enough and the increase in meal size large enough to make high-appetite people usually select it, while low-appetite people are happy to have the smaller ("normal") meal.

    Consider as an example the BigMac. According to Wolt, the price of a BigMac Burger in McDonald's Meilahti on November 12th 2020 is 4.20 euros, while medium fries are 2.30 euros and a medium soda is 2.45. Together, these would then cost 8.95 as single items, but the medium BigMac Meal consisting of these items only costs 7.35 euros, making the savings from buying them as a bundle 1.6 euros.

    Incidentally, with one additional euro, one can upgrade to a large BigMac Meal, containing extra fries and a larger soda. Alone, the larger soda costs 0.30 euros more than the medium soda, and the larger fries cost 0.35 euros more than the normal fries, so once one has selected the bundle, the price discount for fries and soda is actually worse than it would be for each of those items alone.

30. (a) In a simple pricing scheme there is just one price per buffet visit. Both customer types have the same choke price at $p = 24$ and positive demand at lower prices, so we can simply sum the individual demands together to get the aggregate demand $Q^d(p) = 300Q_r^d(p) + 200Q_g^d(p) = 8400 - 350p$, so $P^d(q) = 24 - q/350$. Marginal revenue is then $\partial(P^d(q)q)/\partial q = 24 - 2q/350$. Setting MR equal to marginal cost 4 yields $p^* = 14$ as the optimal simple price.

Taking into account fixed costs, this would result in a profit of $Q^d(p^*)(p^* - 4) - 40000 = (8400 - 350 \times 14)(14 - 4) - 40000 = -5000$. If only simple pricing were available to the restaurant it would not be able to break even and would produce nothing.

(b) With a subscription fee the restaurant can implement a two-part tariff. The two customer types can be ordered according to their demands ($P_r^d(q) = 24 - 2q \leq P_g^d(q) = 24 - q$ for all q) but not segmented (both have to be offered the same fixed monthly fee). In this setting, the firm maximizes its profits by extracting as much consumer surplus as possible using the fixed fee. There are two strategies that might maximize the firm's profit. First is to set the fixed fee so that the consumer surplus of the regulars is fully extracted, in which case it will serve both types but some of gluttons' consumer surplus will remain unextracted. If it sets the fee any lower, it will unnecessarily forgo extracting some surplus from both types. If instead it sets the fee any higher, the regulars will not become customers at all, in which case it might as well set the fee equal to the gluttons' consumer surplus, with unit price equal to marginal cost, which is the second potentially profit-maximizing strategy.

Suppose it follows the first strategy. The monthly fee that fully extracts the regulars' consumer surplus can be written as a function of p as

$$F(p) = \frac{1}{2}((24 - p) \times (12 - 0.5p)) = \frac{288 - 12p - 12p + 0.5p^2}{2} = 0.25p^2 - 12p + 144.$$

This corresponds to the maximum fee that both types of customers - 500 customers in total - will be willing to pay at unit price p. The firm's profit function then becomes:

$$\begin{aligned}
\Pi(p) &= 500F(p) + (p - 4)(300Q_r^d(p) + 200Q_g^d(p)) - 40000 \\
&= 500 \times (0.25p^2 - 12p + 144) + 9800p - 350p^2 - 73600 \\
&= 125p^2 - 6000p + 72000 + 9800p - 350p^2 - 73600 \\
&= 3800p - 225p^2 - 1600
\end{aligned}$$

Maximizing this with respect to p yields the profit maximizing unit price:

$$\begin{aligned}
&\frac{\partial \Pi(p)}{\partial p} = 3800 - 450p = 0 \\
\implies\ & p^* = \frac{380}{45} = \frac{76}{9} = 8 + \frac{4}{9} \approx 8.44
\end{aligned}$$

This gives the firm a profit of $\Pi(76/9) = 3800 \times (76/9) - 225 \times (76/9)^2 - 1600 = 14444 + \frac{4}{9} \approx 14444.44$.

Now suppose the firm follows the second strategy. It will then set $p = MC = 4$, while setting the monthly fee so that it extracts fully the surplus from gluttons: $F^g(4) = \frac{1}{2}((24-4) \times (24-4)) = 200$. Now only gluttons will pay the fee, so this yields a profit of $\Pi^g(p) = 200 \times 200 + (4-4)(200Q^d_g(p)) - 40000 = 0$. Clearly the firm will not want to prize out the regulars.

In the simple prize setting, consumer surplus would be zero for all customers since the firm could not make a profit and would thus not enter the market at all. [8] With the two-part tariff, the consumer surplus for regulars is zero by construction. For gluttons, without the fee the total consumer surplus at $p = 8 + \frac{4}{9}$ would have been $200 \times F^g(8+\frac{4}{9}) = 200 \times \frac{1}{2}((24-(8+\frac{4}{9})) \times (24-(8+\frac{4}{9}))) = 100 \times (\frac{140}{9})^2 \approx 24197.53$. From this we need to subtract the surplus lost due to the fee $200 \times F(8+\frac{4}{9}) = 200 \times \frac{1}{2}((24-(8+\frac{4}{9})) \times (12-0.5(8+\frac{4}{9}))) = 100 \times (\frac{140}{9} \times \frac{70}{9}) = 980000/81 \approx 12098.77$, bringing the total surplus of gluttons to approximately $24197.53 - 12098.77 = 12098.76$. Gluttons and the firm benefit from the two-part tariff, while regulars are left with zero surplus in both cases.

As an additional comment, notice here that we've taken as given that everyone is either a consumer loyalty card holder or not a customer at all. In fact, this is necessary, and can be either interpreted as the firm requiring the customer loyalty card for the buffet serving, or setting the unit price of the buffet without the card so high that everyone will purchase the card. To see why there is no situation where only some customers get the card, consider what happens if we set the unit price without the card, p_{NC} so that that some customers will choose to not buy the card. It's trivial to see that if p_{NC} is low enough that regulars will choose it rather than the subscription tailored towards extracting all surplus from them, then there will be some surplus unextracted from them, and the firm could increase its profit by mandating the card. Suppose instead the firm tailors the subscription fee towards the gluttons, and then tries to set p_{NC} so that regulars will choose it. For the gluttons not to also choose the higher unit price instead of the subscription service, their surplus from it has to be negative (by construction, their surplus is zero from the subscription service), i.e. $F(p_{NC}) = \frac{1}{2}((24-p_{NC}) \times (24-p_{NC})) < 0$. But the left hand side is never negative, and only zero at $p_{NC} = 24$. So the unit price would have to be 24 euros for the gluttons just to be indifferent between it and the subscription service. But that price would make the demand by both gluttons and regulars zero!

(c) Since serving only gluttons was not profitable even with the lower marginal cost, it

[8] One could also imagine a hypothetical world where the firm is somehow locked in to its fixed costs and offers services at $p = 14$ even without making a profit. In that case, total consumer surplus for regulars would have been 9000 and for gluttons 10000.

clearly won't be with the higher marginal cost either - the fee will be smaller while the fixed costs stay the same. Thus the profit-maximizing pricing scheme will follow the same strategy as previously. The fee is also still the same as a function of p, since it is still targeted at extracting the surplus from the regulars.

Notice that since the only thing that changes from 30b is the marginal cost of serving gluttons, we can simply subtract the change in the variable cost (marginal cost times the gluttons' demand) from the original profit function in 30b:

$$\begin{aligned}\Pi^c(p) &= 3800p - 225p^2 - 1600 - 2 \times 200 \times (24 - p) \\ &= 3800p + 400p - 225p^2 - 1600 - 9600 \\ &= -225p^2 + 4200p - 11200.\end{aligned}$$

Maximizing this with respect to p yields the profit maximizing unit price:

$$\frac{\partial \Pi(p)}{\partial p} = 4200 - 450p = 0$$
$$\implies p^* = \frac{28}{3} = 9 + \frac{1}{3} \approx 9.33$$

And the firm gets a profit of $\Pi^c(28/3) = -225 \times (28/3)^2 + 4200 \times (28/3) - 11200 = -19600 + 39200 - 11200 = 8400$.

31. (a) Notice that the health-conscious customers here are the low demand types while low-income customers are the high-demand types. Here "L" refers to "low income" and "H" to "health conscious" types. Notice also that it doesn't matter for the pricing decision whether there are one or one million customers of each type. Let's assume, for convenience, that there is one consumer of each type.

Suppose first that the firm wants to sell both types of gruel. The profit maximizing version prices in this case are such that it sets the price of thin gruel equal to the lowest valuation (the valuation of H-types, 1.2 €) and then sets the price of thick gruel as high as it can be so that the types with the highest valuation (L-types) will select it instead of thin gruel. This is achieved when the price of thick gruel, p, is set so that $1.5 - 1.2 = 2.9 - p \implies p = 2.6$. Such a pricing strategy yields the firm a profit of $2.6 - 2 + 1.2 - 1 = 0.6 + 0.2 = 0.8$.

Now suppose the firm instead sells only thick gruel. If it wants to sell to both types, it should set the price equal to the lower valuation, which would yield a profit of $2 \times (2.3 - 2) = 0.6$ if instead it only sells to the L-types, it should set the price to its valuation, yielding it a profit of $2.9 - 2 = 0.9$. Also note that since the difference between the valuations and marginal cost of thin gruel is smaller than for thick gruel for both types, selling thin gruel exclusively can't be more profitable. Hence, selling only thick gruel to L-types exclusively is the most profitable pricing scheme.

(b) Suppose there are 100 customers, α percent of whom are L-types. Selling only thick gruel at 2.9 euros yields a profit of $100\alpha \times 0.9 + 100(1-\alpha) \times 0$. Selling to both types yields a profit of $100\alpha \times 0.6 + 100(1-\alpha) \times 0.2$. The former strategy will be more profitable than the latter as long as

$$\begin{aligned} 100\alpha \times 0.9 &> 100\alpha \times 0.6 + 100(1-\alpha) \times 0.2 \\ \implies 0.9\alpha &> 0.6\alpha + 0.2 - 0.2\alpha \\ \implies \alpha(0.9 - 0.6 + 0.2) &> 0.2 \\ \implies \alpha &> \frac{0.2}{0.5} = 0.4 \end{aligned}$$

Thus when the H-types make up more than 60% of the market, the firm should switch to selling both types of gruel. Again the actual number of customers did not matter for optimal pricing.

32. (a) First, let's figure out the inverse demands of the types:

- Households: $Q_1(p) = 60 - 12p \implies P_1(q) = 5 - \frac{1}{12}q$
- Industrial customers: $Q_2(p) = 100 - 20p \implies P_2(q) = 5 - \frac{1}{20}q$

Since $P_1(q) < P_2(q)$ for any (positive) q, lets call the industrial customers "high demand customers" and denote them with the subscript H. Conversely, households are "low demand customers" denoted with the subscript L. There are two strategies that might maximize Fish plc's profits: it might sell a small package, q_L, at a price p_L to the low demand customers and a large package, q_H, at a discounted price p_H to high demand customers, or it might sell exclusively to high demand customers.

Suppose it follows the first strategy. Recall that in an optimal quantity discount scheme there is "no distortion at the top", meaning that the larger package that Fish plc sells will be of a socially efficient size. This happens when $P_H(q_H) = MC$. Since $MC = 0.5$, this means that $5 - \frac{1}{20}q_H = 0.5 \implies q_H = 20 \times (5 - 0.5) = 90$.

There is also "no consumer surplus at the bottom", meaning that the smaller package will be sold to low demand types at reservation value: $p_L = B_L(q_L)$, where $B_L(q_L)$ is the reservation value of the low demand customers for the smaller package. Since the demand is linear, the reservation value for a package of size q is the area of the trapezoid under the demand line from 0 to q: $B_L(q) = \frac{5+P_L(q)}{2}q = 5q - \frac{1}{24}q^2$.

Finally, the price of the larger package must be just low enough so that high demand customers will still choose it over the smaller package. Their surplus from the larger package then equals the surplus form the smaller package: $B_H(q_L) - p_L = B_H(q_H) - p_H \implies p_H = B_H(90) - B_H(q_L) + B_L(q_L)$. Again, we get the reservation value for the high demand customers, $B_H(q)$, from the area of the trapezoid under the high demand line: $B_H(q) = \frac{5+P_H(q)}{2} q = 5q - \frac{1}{40}q^2$. The price of the larger package is then:

$$\begin{aligned}
p_H(q_L) &= B_H(90) - B_H(q_L) + B_L(q_L) \\
&= 5 \times 90 - \frac{1}{40} \times 90^2 - 5q_L + \frac{1}{40}q_L^2 + 5q_L - \frac{1}{24}q_L^2 \\
&= 450 - 202.5 + \frac{1}{40}q_L^2 - \frac{1}{24}q_L^2 \\
&= 247.5 + \left(\frac{1}{40} - \frac{1}{24}\right) q_L^2 \\
&= 247.5 - \frac{1}{60}q_L^2
\end{aligned}$$

Additional intuition for the above is that the high demand types will get to keep some of their reservation value ($B_H(90) = 247.5$) as an "information rent" that incentivizes them to self select into the larger package. How much information rent they get depends on how large the smaller package is.

Let's measure the numbers of customers and thereby profits in thousands, so that $N_L = 1$ and $N_H = 2$. Now Fish plc's profit can be expressed as a function of the size of the smaller package alone:

$$\begin{aligned}
\Pi(q_L) &= 2p_H + p_L - (2q_H + q_L) \times \mathrm{MC} - 250 \\
&= 2 \times \left(247.5 - \frac{1}{60}q_L^2\right) + \left(5q_L - \frac{1}{24}q_L^2\right) - (2 \times 90 + q_L) \times 0.5 - 250 \\
&= \left(495 - \frac{q_L^2}{30}\right) + \left(5q_L - \frac{q_L^2}{24}\right) - 0.5q_L - 340 \\
&= -0.075q_L^2 + 4.5q_L + 155
\end{aligned}$$

Maximizing this with respect to q_L yields the first order condition $4.5 - 0.15q_L = 0 \implies q_L = 30$. Thus, the size of the smaller package is 30 kg. Now we get the price of the small package from $p_L = B_L(q_L) = 5 \times 30 - (1/24) \times 30^2 = 150 - 37.5 = 112.5$ and the price of the larger package from $p_H = 247.5 - (1/60) \times 30^2 = 247.5 - 15 = 232.5$. The profit-maximizing quantity discount gives Fish plc profits of $\Pi(30) = -0.075 \times 30^2 + 4.5 \times 30 + 155 = 222.5$. thousand euros.

Suppose the firm instead sells only to the high demand customers. Its profit is then maximized when it sells the efficient package at their reservation value: $p_H = B_H(90) = 247.5$. This gives the firm a profit of $\Pi(0) = 2 \times (247.5 - 0.5 \times 90) - 250 = 155$ thousand euros, which is less than from selling to both types of customers. Hence the quantity discount is indeed the profit maximizing scheme.

(b) The middlemen could buy the the larger package of $q_H = 90$ kilograms for a cost of$p_H = 232.5$. They could then split this package to three small packages of $q_L = 30$ kilos and sell them at $p_L = 112.5$ euros a piece for households. This would yield them a profit of $3 \times 112.5 - 232.5 = 337.5 - 232.5 = 105$ euros per package. Since there are a 1000 households and 2000 industrial customers, assuming the middlemen were fast enough actors, they could buy 333 of the large packages intended for the industrial customers and sell the split packages to 999 households for a total profit of $333 \times 105 \approx 35$ thousand euros (it would not be worth it selling to the 1000th household, as they would have to buy a full additional large pack and only be able to sell a third of it - this would yield them a loss of $232.5 - 112.5 = 120$ euros for that package).

33. For the purpose of the optimal pricing it is convenient to transform the customer reservation values into values net of marginal cost. Let's also add the net valuations for the bundle in the same table. Since the goods are neither substitutes or complements, the customer valuation for a bundle is simply the sum of the valuations for the two goods.

€	Grouse	Pineapple	Bundle
Bourgeois	10	10	20
Students	1	8	9
Workers	9	6	15

As there is an equal amount of each type, to simplify calculations, let's assume for now that there is one of each. The absolute number of customers does not affect the relative profitability of various pricing strategies when the marginal costs are constant for each good.

(a) For each good there are three possible price points that correspond to selling to one, two or three customer types. Consider first the pricing of grouse. By selling to all customers profits are $3 \times 1 = 3$, by selling to two highest-value types profits are $2 \times 9 = 18$, and by only selling to highest-value types profits are merely 1×10.

Similarly, for pineapple, the comparison is between $3 \times 6 = 18$, $2 \times 8 = 16$, and 1×10, of which selling to all three types is the best.

As there are 100 customers of each type, maximized profits are $100 \times (18 + 18) = 3600$ euros. Adding back the MCs to optimal net prices yields the actual optimal "list prices" as $9 + 5 = 14$ euros for grouse and $6 + 3 = 9$ euros for pineapple.

(b) Under pure bundling only the bundle is sold and priced using basic pricing. Just like in part 33a, let's compare profits at the three relevant price points: $3 \times 9 = 27$, $2 \times 15 = 30$, and 1×20. Maximized total profits are $100 \times 30 = 3000$ euros (worse than basic!). Optimal price for the pure bundle includes the marginal costs: $15 + 5 + 3 = 23$ euros.

(c) With mixed bundling, Acme can allow either grouse, pineapple, or both to be bought as individual items separately from the bundle. If both are sold then the sum of prices is more than the price of the bundle. Just like under basic pricing, any deal that is on sale must be just at the borderline of inducing a profitable sale to one of the customer types.

Let's first depict all three types in "type space", where each axes represents the net valuations for one good, see Figure 28.

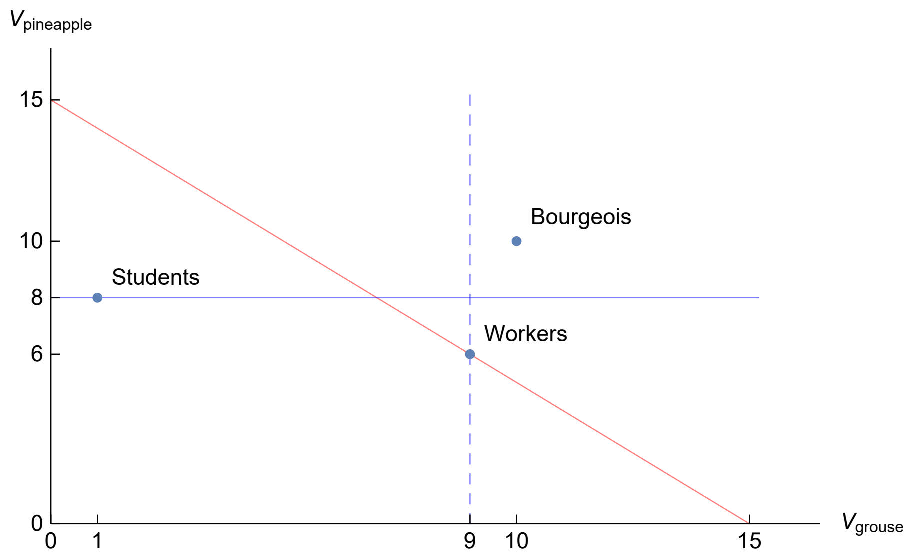


Figure 28: Consumer types in net valuation space.

It is apparent that students are the type with a relatively high value for pineapple, hence, if pineapple is to be sold separately then its price will be determined by the student value 8. Similarly, if grouse is to be sold separately its price will be determined by worker value at 9. The bourgeois have a higher valuation for both goods than other types. This means that their valuations cannot be binding if Acme is to use a mixed bundling strategy. No matter what strategy is used, the bourgeois will buy the bundle.

If students are the type that will only buy one good then the bundle price is determined by workers, and profits are $8 + 2 \times 15 = 38$. (Acme could at the same time also sell grouse at any price that exceeds the worker value 9, but there would not be much point as no one would buy it.) If workers were the type that is only sold one good then the bundle price would be determined by students, but at their bundle valuation 9 workers would also buy the bundle, so that would just amount to selling the bundle to all types (for a profit of only $3 \times 9 = 27$).

The optimal mixed bundling strategy earns total profits of $100 \times 38 = 3800$ euros. The list prices are $8 + 3 = 11$ euros for pineapples and $15 + 3 + 5 = 23$ euros for the bundle. Mixed bundling is the most profitable pricing strategy considered here. It can never do worse than basic pricing or pure bundling because it includes both as special cases.

34. (a) The welfare (profit) of an individual tuna fisher is given by $v(n) = 2x(n) - 20$. The efficient number of fishing boats maximizes the total welfare, i.e. the number of fishers times their individual welfares:

$$\begin{aligned} W(n) &= nv(n) \\ &= n(2 \times (80 - 0.2n) - 20) \\ &= 140n - 0.4n^2 \end{aligned}$$

Maximizing the above with respect to n gives the first order condition $140 - 0.8n = 0 \implies n_{SE} = 175$. This would give each fisher a catch of $x(175) = 80 - 0.2 \times 175 = 45$, yielding a profit of $v(175) = 2 \times 45 - 20 = 70$. The total profit from the total catch $(175 \times 45 = 7875)$ tons is $175 \times 70 = 12250$ monetary units.

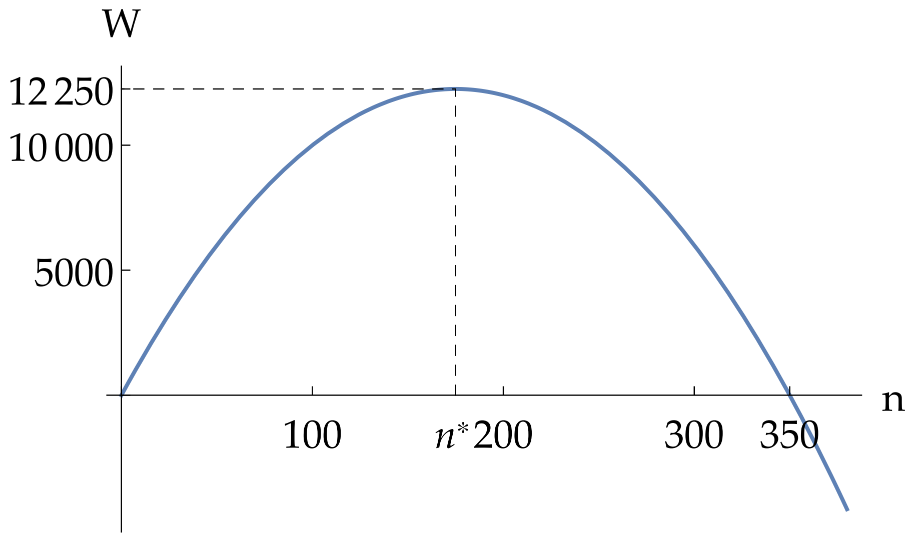


Figure 29: Total welfare as a function of the number of fishing boats in part 34a.

(b) Without restrictions, tuna boats will enter until the welfare of the next entrant falls below zero. That is, another tuna fisher will enter as long as:

$$\begin{aligned} v(n) = 2x(n) - 20 &\geq 0 \\ \implies 160 - 0.4n - 20 &\geq 0 \\ \implies 0.4n &\leq 140 \\ \implies n_{EQ} &\leq 350 \end{aligned}$$

meaning that 350 tuna fishing boats will enter without restrictions on their entry. The per-boat tuna catch in this case is $x(350) = 80 - 0.2 \times 350 = 10$, meaning that the total catch is $350 \times 10 = 3500$ tons. The profits, per boat and in total, are by definition zero. Hence, without restrictions, the total catch drops to less than a half of the efficient case, while the profits drop to zero.

(c) We know from 34a that total welfare is maximized when $n = 175$. The market price of the license, p, will be equal to the benefit for the nth entrant, i.e. $p = v(n)$. Again, we know from 34a that $v(175) = 70$. Hence, market price in the socially efficient case is $p_{SE} = v(175) = 70$.

35. (a) No one will choose a slower road voluntarily, so in equilibrium either the travel time is the same on both roads or all drivers use the same road. Here the Expressway is certainly faster at 30 minutes before congestion kicks in at 5000 drivers, but would be

slower than the Highway if everyone used the Expressway, so there will in equilibrium be drivers on both roads. The Highway is at its fastest at 45 minutes as long as it gets no more than 500 drivers, at which point the Expressway would take $T_1(10000-500) = 30 + 4500/50 = 120$ minutes. Therefore in equilibrium both roads will be congested, which means that between 5000 and 9500 drivers take the Expressway. Travel times are equal if n drivers take the Expressway, the remaining $10000-n$ the Highway. Therefore

$$\begin{aligned} 30 + (n-5000)/50 &= 45 + (10000-n-500)/100 \implies \\ 30 + n/50 - 100 &= 45 - n/100 + 95 \implies \\ n = 7000 & \end{aligned}$$

choose the Expressway and remaining 3000 the Highway. This calculation amounted to finding the crossing point of two travel times functions, shown if Figure 30, but note that for this to work both travel times must be written as a function of the same variable. Equilibrium travel time on both roads and therefore also the average travel time is $T_1(7000) = T_2(3000) = 70$ minutes.

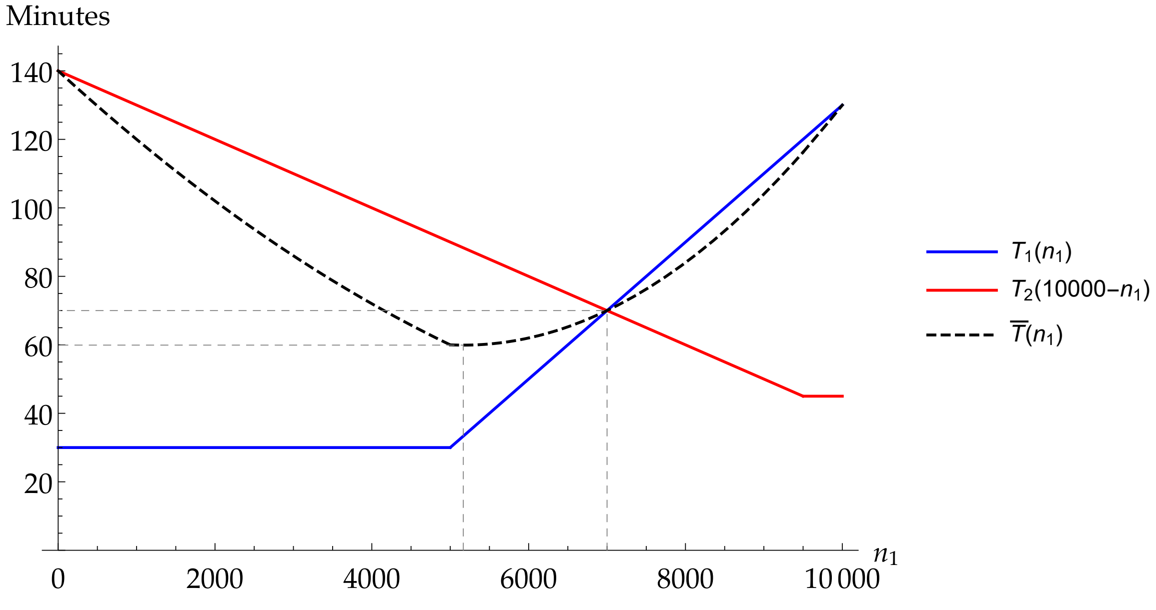


Figure 30: Travel times as a function of n_1, the number of drivers on the Expressway.

(b) Here the maximization of welfare amounts to minimization of total (and average) travel time. Let's set up the total travel time as a function of the number of drivers on Expressway. Let's again use n to denote the drivers on the Expressway. If $n < 5000$ then the Expressway is faster than the Highway but not congested, so clearly optimal n will be above 5000. I.e., if $n < 5000$ it is possible to shift drivers from Highway to Expressway without slowing down the Expressway at all. Since at the optimum both roads will be congested we can write total travel time as

$$\begin{aligned} T(n) &= nT_1(n) + (10000-n)T_2(10000-n) \\ &= n\left(30 + \frac{n-5000}{50}\right) + (10000-n)\left(45 + \frac{10000-n-500}{100}\right) \\ &= n\left(\frac{1}{50}n - 70\right) + (10000-n)\left(140 - \frac{1}{100}n\right) \\ &= \frac{3}{100}n^2 - 310n + 1400000 \end{aligned}$$

The first order condition is

$$\frac{6n}{100} - 310 = 0 \implies n^* = \frac{31000}{6} \approx 5167.$$

As T is an upwards opening parabola this is indeed the minimizer. The resulting average travel time is $\bar{T}^* = T(5167)/5167 \approx 60$ minutes. Average travel time is depicted as the dashed curve in Figure 30. Note that at optimum the Expressway is faster, at $T_1(n^*) = 33\,1/3$ minutes while the Highway takes $T_2(10000-n^*) = 88\,1/3$ minutes.

A welfare-maximizing road pricing scheme must incentivize the right amount of drivers to choose the Highway even while it is slower by $T_2(n^*) - T_1(n^*) = 55$ minutes. Given that the drivers value saved time at €0.2/minutes, there must be a toll of $55 \times 0.2 = 11$ euros on the Expressway.[9] This toll makes the drivers indifferent between the two roads, and if too many drivers were taking the Expressway then the saved time there would no longer be worth the toll.

The toll increases welfare by reducing average travel time by $70 - 60 = 10$ minutes, which is worth $10000 \times 10 \times 0.2 = 20$ thousand euros to the drivers. This is also the impact of the road pricing on total welfare. The drivers will pay in total $n^* \times 11 \approx 56.8$ thousand euros of tolls, which does not affect total welfare but it is a pure transfer from drivers to the government.

As an aside, the welfare-enhancing toll is a deadweight-loss-free source of revenue for the government, so in principle it enables the reduction of some welfare-loss inducing tax elsewhere in the economy.

(c) All low income drivers will choose the Expressway as it is faster and costs the same. High income drivers, on the other hand, will choose the Highway only as long as the time saved there is worth the €11 toll, which is the case when the time saving is 55 minutes. Hence again the total number of drivers on the Expressway will be unchanged from part 35b. Hence, in equilibrium the Highway is fully populated by high income drivers and Expressway has all the low income drivers and $n^* - 5000 \approx 167$ high income drivers. Average travel time is the same for everyone as in part 35b, but many fewer drivers are paying the toll. Total welfare is unaffected, but the transfer from drivers to

[9] More generally, any combination of tolls where the Expressway is more expensive by 11 euros works here.

the government is smaller. It is now only $167 \times 11 \approx 1.8$ thousand euros, about €50k less than before.

In part 35b it was not determined who pays the toll and who takes the Highway, because every driver had the same level of welfare. Now the low-income drivers are better off by the amount of the toll. Relative to part 35b, they get a transfer of $5000 \times 11 = 55$k euros from the government. High income drivers' welfare is not affected, they are still indifferent between taking the Expressway + paying the toll and taking the Highway.

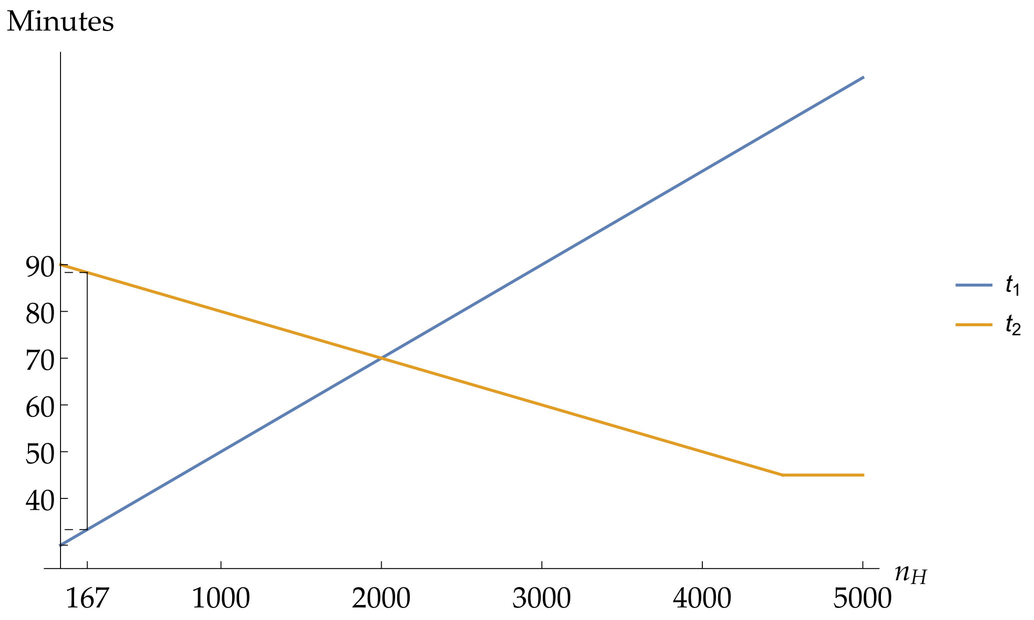


Figure 31: Travel times as a function of the number of high-income drivers on the Expressway in part 35c.

36. (a) Denote the membership fee by F. With a constant $z = 1$, everyone in Lintukoto will be a customer if $v(n) = \sqrt{n} \geq F$ and otherwise no-one. This means that profit is maximized when $F = \sqrt{n}$. Since serving another customer does not cost anything to AllCaps, profits are maximized when everyone is a customer, i.e., when $F = \sqrt{10000} = 100$.

(b) Since there are 10 000 people, with the lowest z at 0, the highest at 2 and an equal distance between each, the second highest$z$ is $2 - \frac{2-0}{10000} = 2 - \frac{2}{10000}$, the third highest is $2 - 2 \times \frac{2}{10000}$ etc. In general, the preference parameter for the individual with the ith highest preference (starting the count from 0) is $z_i = 2 - \frac{i}{5000}$. Notice that with any fee that attracts some customers but not others, it will be the customers with the higher valuations (i.e. the higher zetas) that join the network. With n users, the lowest valuation included is $v(n) = (2 - \frac{n}{5000})\sqrt{n}$. The fee that gets n users to join is equal to the lowest valuation in that group, i.e. $F = v(n)$, and the revenue generated is

$$\begin{aligned} R(n) &= nF = nv(n) = n(2 - \frac{n}{5000})\sqrt{n} \\ &= 2n^{1.5} - \frac{n^{2.5}}{5000} \end{aligned}$$

Marginal cost is zero, so profit maximization amounts to maximizing revenue. The first order condition is

$$3\sqrt{n} - \frac{2.5}{5000}n^{1.5} = 0 \implies \sqrt{n}(3 - \frac{2.5}{5000}n) = 0,$$

which is fulfilled either when $n = 0$ or $3 - \frac{2.5}{5000}n = 0 \implies n = 6000$. The latter is clearly the maximum, as $R(0) = 0$. With $n^* = 6000$, the lowest valuation, which equals the fee, is $v(n^*) = (2 - \frac{6000}{5000})\sqrt{6000} = 0.8 \times \sqrt{6000} \approx 61.97$ euros.

(c) Once FreeRant has enough customers AllCaps can no longer compete with price. Customers will find FreeRant preferable if AllCaps network has no more than n users, such that

$$\sqrt{n} < 2\sqrt{10000 - n} \implies n < 4 \times (10000 - n) \implies 5n < 40000 \implies n \leq 8000.$$

This is the tipping point: once FreeRant has attracted at least 2000 customers AllCaps can no longer survive in equilibrium. FreeRant can always match its subscription price and get all the 10 000 customers to itself.

Given that AllCaps has a fixed cost, in principle it could happen that it would be driven out of business before n reaches the tipping point. However, that is not the case here, as at the tipping point AllCaps still earns $8000v(8000) \approx 716$k euros, much above the fixed cost 200k.

37. A timely example of a network good is the just-released PlayStation 5 gaming system, currently (November 23rd, 2020) sitting at 529.90 euros (though temporarily sold out) at Verkkokauppa.com. While it has value even without a network, its value certainly increases as more users buy the system, due to many games these days having some form of online multiplayer capability as an essential part of the experience. The PlayStation—and other gaming systems—actually have many types of network externalities, making them an interesting case study. First, network externalities operate mainly through the games released for the system rather than the system itself, and for each game with an online multiplayer mode, there is a critical mass of players that make it possible for any one user to start playing at any given time and be sure that there is someone to play with.

In addition to this sort of critical mass-effect, there is another network externality for games that one might want to play with friends rather than random strangers. In these cases, the game becomes more desirable when more people in one's own social network play it. This can also affect the desirability of the system itself: if all of your friends play a certain game on Xbox, you may switch from PlayStation to Xbox (or, buy both if you have the money) even if that same game would be available for PlayStation, because you can't play online with them using the PlayStation version. While these days it is possible to play some games online with players using different gaming systems, this is a relatively recent development at least for the PlayStation (according to Wikipedia, Sony only started to allow game developers to support cross-platform play in October 2019 after much outrage from their customers), and platform-exclusive games are still a major selling point for gaming systems. It will be interesting to see how this develops in the future—perhaps the gaming system manufacturers have noticed that its more profitable to have one big network with all their competitors, even if they get a smaller share of the revenue generated by that, than to try and have exclusive competing networks (or perhaps they are just protecting their image).

Finally, network effects are an important part of how gaming system manufacturers are able to sell a new generation of systems to people who already own a previous version. From my (limited) understanding, the PlayStation 5, for example, is not that much more impressive technology-wise than the PlayStation 4 (at least in comparison to differences between previous generations), but by releasing some new games only on the new system, Sony will lure some players away from their old system, decreasing the network externality on it and increasing it on the new system, making it more likely that more people will switch over to the new system. Although apparently the PlayStation 5 includes backwards compatibility so that players will be able to play PlayStation 4 games online with players who are still using PlayStation 4, this obviously does not apply to games released only for the 5th generation system, which will be more and more every year. Furthermore, backwards compatibility does not apply to older generations, and perhaps the plan is that when the inevitable PlayStation 6 comes out, backwards compatibility will again go back just one generation.

38. (a) If all four types of boats are sold, the buyers' expected value from buying a boat is $EV_4 = \frac{1}{4} \times (20 + 24 + 28 + 36) = 27$. But this is less than sellers' valuation for the perfect boats, and hence only three types of boats are traded. But this means that the buyers' expected values is only $EV_3 = \frac{1}{3} \times (20 + 24 + 28) = 24$, which is less than the sellers' valuation for good boats, which in turn means that only two types of boats are sold, leaving the buyers with an expected value of $EV_2 = \frac{1}{2}(20 + 24) = 22$. Since $EV_2 > 20$, the market doesn't unravel further, meaning that two types of boats, junk boats and fine boats are traded. The total number of boats traded is $2 \times 1000/4 = 500$. Figure 32 shows this situation graphically, with the expected values of buyers when boat qualities up to a given type are on the market plotted with golden circles, and the sellers' valuations for the corresponding type plotted in blue.

Since buyers value each boat type more than sellers, it would be efficient to trade all the boats. With symmetric information, the sellers would sell the boats at the buyers' valuations. This leaves all sellers better off by the difference in the valuations between them and the buyers, increasing total welfare by $250 \times (5+4+3+4) = 4000$ €k. With asymmetric information, only the fine and junk boat owners get to sell, and both sell at $EV_2 = 22$. This leaves junk sellers better off by 7, but also junk buyers worse off by 2. Both fine sellers and buyers meanwhile are left better off by 2. Thus total welfare increases by $250 \times (7 - 2 + 2 + 2) = 2250$ k€.

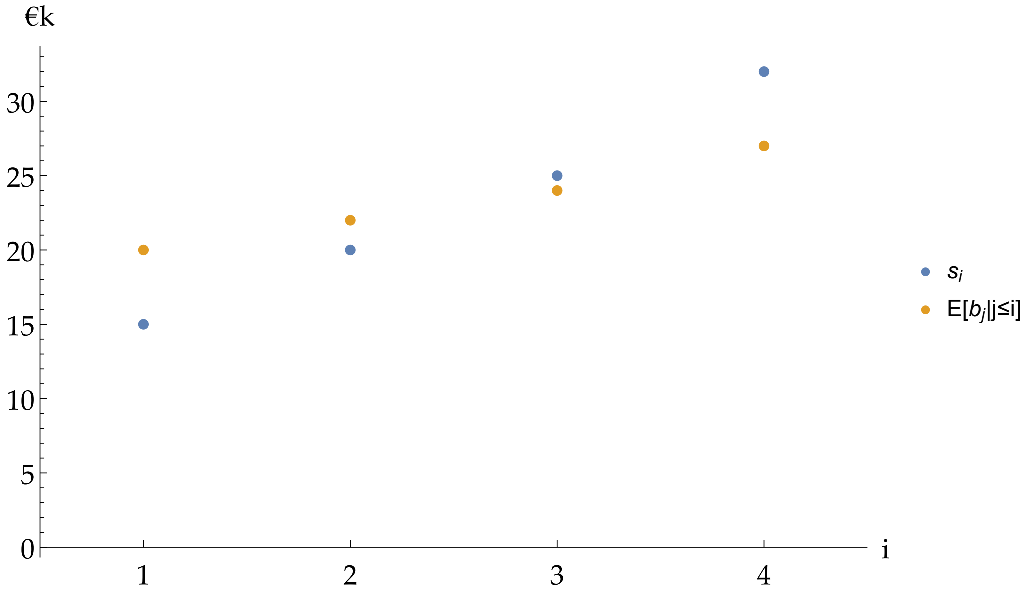


Figure 32: Valuation by seller type i is s_i, and expected buyer value if i is highest quality seller in the market is $E[b_j|j \leq i]$.

(b) The dealer makes its profit by buying the boats at the sellers valuation or the price that the seller would get in the market, whichever is higher, and selling it forward to the customer at their valuation after credibly disclosing its quality. Note that if it didn't verify and disclose the quality of each boat it deals, it would just be buying and selling them at the market price and making zero profits. Since the dealer will verify the quality of a boat, it can make a contract with the seller where the price it pays for the boat is conditional on the quality. Also, note that the price the sellers can get in the market depends on what type of boats the dealer deals, since the types of boats dealt by the dealer are effectively out of the market.

Notice first that the dealer can never make a profit by dealing junk boats: the price their sellers get in the market is always at least equal to the buyer valuation.

Suppose then it only deals in perfect boats. Since the perfect boat sellers are not able to sell in the market, the dealer only needs to pay them their valuation, yielding it a profit of $36 - 32 - 2 = 2$ k€ per boat. Suppose next the dealer adds good boats to its repertoire. The good boat sellers are also unable to sell in the market with or without the dealer dealing in perfect boats. Hence, the dealer will make a profit of $28 - 25 - 2 = 1$ per good boat. Since this doesn't affect it's ability to deal in perfect boats - the expected value of a boat in the market is still less than the perfect boat seller's valuation even if good boats are taken out of the mix - it should do both.

Should it also deal in fine boats? With only fine and junk boats in the market, the market price of 22 is still higher than the valuation of the fine boat sellers, so the dealer will need to pay that. This means that it will make $42 - 22 - 2 = 0$ €k profit from dealing in fine boats as well (obviously this doesn't affect its ability to deal in good or perfect boats because the market price would drop to the buyers' valuation of junk boats), making it indifferent between dealing and not dealing in them. Hence, there are two possibilities with different implications for total welfare:[10]

(i) The dealer trades only in perfect and good boats, in which case they will sell at valuation while losing the verifying cost, while the payoffs for junk and fine types will be exactly as in the asymmetric information case of 38a. Taking into account the cost of verification, this yields a total welfare of $250 \times (4 - 2 + 3 - 2) + 2250 = 3000$k€.

(ii) The dealer trades perfect, good and fine boats, leading to the symmetric information -situation from 38a, expect now 2k€ per boat is lost for perfect, good and fine boats, yielding a total welfare of $4000 - 750 \times 2 = 2500$. Hence, the former equilibrium yields a higher total welfare.

(c) As long as the shares of junk and fine boats are equal, the expected value from buying in a market where only they are traded is the same. Hence, the market will never fully unravel regardless of the share of perfect types. However, if the share of perfect boats is high enough, there will be no unraveling at all, as the expected buyer value over all types then exceeds the highest seller valuation. With fraction x perfect types, the other three types will each have $(1 - x)/3$ of the total.

$$EV(x) = 36x + \frac{1-x}{3} \times (28 + 24 + 20) \quad \geq 32$$

$$\implies 24 + 12x \quad \geq 0 \quad \implies x \geq \frac{2}{3}$$

Thus there is no unraveling if at least 2/3 of boats are of perfect quality.

39. (a) If there are no engravings to signal the skill of the watchmaker, the buyers' expected value from buying a watch is $\frac{1}{2} \times (100 + 40) = 70$. This means that both types of watchmakers will make a profit of $70 - 15 = 55$ ducats per watch. Suppose then that high skilled watchmakers engrave their watches, but low-skilled do not, and the buyers know this. Then high skilled watchmakers get a payoff of $100 - 25 - 15 = 60$, which is higher than from not signaling. This leaves the low-skilled with a payoff of $40 - 15 = 25$, which is clearly more than if they tried to signal as well - this would yield them a payoff of $40 - 75 - 15 = -50$ per watch. As the buyers know what the watchmakers' payoffs are, the signal is successful at separating the types, and because it is profitable for the high skilled watchmakers, they will use it.

[10] Either one of these is a fully acceptable answer.

(b) Assuming it was common knowledge that such a technology existed, engraving would lose its effectiveness as a signal: it would not matter for the watchmaker's payoffs whether they engrave or not, and the buyers would not be able to distinguish between durable and non-durable watches via engraving. This would mean a return to the non-signal situation of 39a. In this state, the consumer surplus per durable clock would be $100-70 = 30$ and per non-durable clock $40-70 = -30$, while the producer surplus would be 55 for both, yielding a total surplus with x clocks of $\frac{1}{2} \times (55+30+55-30)x = 55x$. Meanwhile, the consumer surplus in the signaling case would be 0 for both clock types, while the producer surplus would be 60 per clock for high skilled and 25 for low skilled watchmakers, yielding a total surplus with x clocks of $\frac{1}{2} \times (60+25)x = 42.5x$, meaning that total surplus would increase by $(55x - 42.5x)/42.5x \approx 29\%$ if someone invents a costless engraving method.

40. (a) The efficient level of spending minimizes expected loss $\mathbb{E}L(x, v) = p(x)v + x$, where $x \in \{0, 1, 2, 4\}$ is a possible level of spending, $p(x)$ is the associated probability of total loss, and v is the level of total loss (i.e., the value of the ship and the cargo). Plugging in the possible levels of spending yields:

$$\begin{aligned}
\mathbb{E}L(0, v) &= 0.2v \\
\mathbb{E}L(1, v) &= 0.08v + 1 \\
\mathbb{E}L(2, v) &= 0.04v + 2 \\
\mathbb{E}L(4, v) &= 0.01v + 4
\end{aligned}$$

Evaluating these for a low value ship ($v = 20$) yields the expected losses $\{4, \underline{2.6}, 2.8, 4.2\}$. Likewise, for a high-value ship ($v = 100$) the expected losses are $\{20, 9, 6, \underline{5}\}$. The optimal level of safety spending is €1 million for a low-value ship, and €4 million for a high-value ship.

Minimizing expected loss is, of course, equivalent with maximizing expected profits $\mathbb{E}\Pi(x, v) = v - L(x, v)$ and would lead to the same conclusions.

(b) With an insurance plan with a coinsurance rate r the expected loss is $\mathbb{E}L_I(x, v, r) = p(x)vr + x$. The insurance premium is a sunk cost from the point of view of the insurees, and can therefore be ignored in their choice of safety spending. Note that spending $x = 0$ on safety is not an options, since Acme requires and can verify that the first million be speng. Plugging in the possible levels of spending and Acme's coinsurance rate $r = 0.35$, we get:

$$\begin{aligned}
\mathbb{E}L_I(1, v, 0.35) &= 0.028v + 1 \\
\mathbb{E}L_I(2, v, 0.35) &= 0.014v + 2 \\
\mathbb{E}L_I(4, v, 0.35) &= 0.0035v + 4
\end{aligned}$$

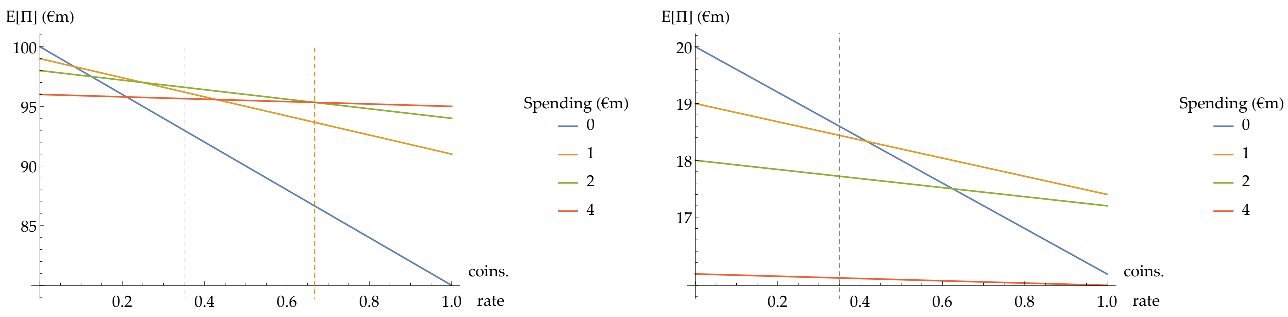
Evaluating these at the two ship values yields expected losses of $\{\underline{1.56}, 2.28, 4.07\}$ for low-value and $\{3.8, \underline{3.4}, 4.35\}$ for high-value ships. With these coinsurance rates low-value shipowners will spend the verifiable €1 million, which suffices for efficiency. Owners of high-value shipowners spend less than the efficient amount, €2 million.

An actuarially fair insurance charges the expected value of payouts. For a low-value ship it is $0.08 \times (1-0.35) \times 20 = 1.04$ million, and for a high-value ship $0.04 \times (1-0.35) \times 100 = 2.6$ million.

As a side note, if the shipowners are risk neutral, they would not benefit even from actuarially fair insurance. For this question risk neutrality was a mathematical simplification, but in practice there exist also regulatory requirements for obtaining insurance coverage.

(c) It is useful to notice that the incentive to spend on safety is increasing in the coinsurance rate as well as in the value of the ship. Also it is never a worry that an insuree would spend too much on safety—the whole problem of insufficient unverifiable safety spending is a moral hazard problem caused by insurance.

For low-value shipowners the verifiable spending i.e. the "first million", is the efficient level, so any coinsurance rate including zero will do. We saw in part 40b that the high-value shipowners spend at t second higest level (€2m) at coinsurance rate $r = 0.35$, so the only question is which rate $r > 0.35$ (if any) is sufficiently high to motivate them to spend €4m instead. In terms of the expected loss, the question is then which r is high enough to make the following inequality true: $\mathbb{E}L_I(4, 100, r) \geq \mathbb{E}L_I(3, 100, r)$. The threshold r is found by solving the associated equality: $0.01 \times 100r + 4 = 0.04 \times 100r + 2 \implies r = 2/3$. A coinsurance rate of 66.7 would be needed for high value ship owners to spend enough on safety.



(a) $v = 100$ **(b)** $v = 20$

Figure 33: Expected profits at various levels of safety spending as a function of the coinsurance rate.

(d) The question is which shipowner value $\bar{v}$ would be high enough to guarantee an efficient level of safety spending at all values greater than $\bar{v}$ when coinsurance rate is $r = 0.35$. We already saw that $v = 100$ was not high enough. Since both optimal and voluntary safety spending are increasing in v, the threshold case will have $x = 4$ as the optimal level. The binding constraint is that a shipowner with value $\bar{v}$ finds it just optimal to spend $x = 4$ rather than the next highest $x = 2$. In other words, the inequality $\mathbb{E}L_I(4, v, 0.35) \geq \mathbb{E}L_I(3, v, 0.35)$ will hold as an equality at $\bar{v}$. Plugging in the definitions this amounts to $0.01 \times 0.35\bar{v} + 4 = 0.04 \times 0.35\bar{v} + 2 \implies \bar{v} \approx 190$ €m.