Fourier analysis (MS-C1420)

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Classification of signals

In this course, we divide signals into two classes:

$$\begin{array}{llll} (A) & \textbf{Analog} & s:\mathbb{R}\to\mathbb{C} & (\text{continuous time } t\in\mathbb{R}), \\ (D) & \textbf{Digital} & s:\mathbb{Z}\to\mathbb{C} & (\text{discrete time } t\in\mathbb{Z}). \end{array}$$

Moreover, we split these classes into two parts: a signal can be

$$\begin{array}{rll} \textit{either} & (0) & \textbf{non-periodic} \\ \textit{or} & (1) & \textbf{periodic}: \; s(t+P)=s(t). \end{array}$$

We shall study the connections between cases
$(A0),(A1),(D0),(D1)$.
Fourier methods in this course:

$$\begin{array}{rr} \textit{Fourier integrals } (A0), & \textit{Fourier coefficients } (A1), \\ \textit{Fourier series } (D0), & \textit{DFT or FFT } (D1). \end{array}$$

Examples: sound, pictures, video; physical measurements; technology and sciences (1-dimensional signals in these notes).

Reminder: operations with complex numbers

Identify the point $(x, y) \in \mathbb{R} \times \mathbb{R}$ in plane
and the complex number $x + \mathrm{i}y \in \mathbb{C}$, where i is the imaginary unit.
Interpretation: real number $x \in \mathbb{R}$ is same as $x + \mathrm{i}0 \in \mathbb{C}$.

$$\begin{aligned}
\textit{Real part} \quad & \mathrm{Re}(x + \mathrm{i}y) := x \in \mathbb{R}. \\
\textit{Imaginary part} \quad & \mathrm{Im}(x + \mathrm{i}y) := y \in \mathbb{R}. \\
\textit{Complex conjugate} \quad & (x + \mathrm{i}y)^* = \overline{x + \mathrm{i}y} := x - \mathrm{i}y \in \mathbb{C}. \\
\textit{Absolute value} \quad & |x + \mathrm{i}y| := (x^2 + y^2)^{1/2} \in \mathbb{R}^+.
\end{aligned}$$

Operations: e.g. $-(a + \mathrm{i}b) := -a + \mathrm{i}(-b)$ and

$$\begin{aligned}
(a + \mathrm{i}b) + (x + \mathrm{i}y) &:= (a + x) + \mathrm{i}(b + y), \\
(a + \mathrm{i}b)(x + \mathrm{i}y) &:= (ax - by) + \mathrm{i}(ay + bx),
\end{aligned}$$

especially $\mathrm{i}^2 = (0 + \mathrm{i}1)^2 = (0 + \mathrm{i}1)(0 + \mathrm{i}1) = -1$.
Euler's formula $\mathrm{e}^{\mathrm{i}t} = \cos(t) + \mathrm{i}\sin(t)$,
and then $\mathrm{e}^{\mathrm{i}(\alpha+\beta)} = \mathrm{e}^{\mathrm{i}\alpha}\mathrm{e}^{\mathrm{i}\beta}$.

Analog non-periodic world (A0)

Continuous time ($t \in \mathbb{R}$) signal s is a "nice-enough" function $s : \mathbb{R} \to \mathbb{C}$. For example, $t \in \mathbb{R}$ time (or position), and $s(t) \in \mathbb{C}$ pressure/temperature/luminosity/position/wave function...
Signal s has **energy density** $|s|^2 : \mathbb{R} \to [0, \infty]$, meaning that

$$\int_{[a,b]} |s(t)|^2 \,\mathrm{d}t \quad \in \quad [0, \infty] \tag{1}$$

is the energy during interval $[a, b] \subset \mathbb{R}$. The **energy** of signal s is

$$\|s\|^2 := \int_{\mathbb{R}} |s(t)|^2 \,\mathrm{d}t. \tag{2}$$

Fourier (integral) transform of a "nice enough" signal $s : \mathbb{R} \to \mathbb{C}$ is signal $\widehat{s} : \mathbb{R} \to \mathbb{C}$ of **frequency** variable $\nu \in \mathbb{R}$, where

$$\widehat{s}(\nu) := \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi t \cdot \nu} \, s(t) \,\mathrm{d}t. \tag{3}$$

Absolute integrability $\|s\|_{L^1} := \int_{\mathbb{R}} |s(t)| \,\mathrm{d}t < \infty$ is "nice enough", because $|\widehat{s}(\nu)| \leq \int_{\mathbb{R}} |s(t)| \,\mathrm{d}t$. Write $s \in L^1(\mathbb{R})$ if $\|s\|_{L^1} < \infty$.

Example. Let $|c| = 1$ for $c \in \mathbb{C}$, and let $s : \mathbb{R} \to \mathbb{C}$, where

$$s(t) := \begin{cases} c\, 2\pi\varepsilon\, \mathrm{e}^{-2\pi\varepsilon\,(t-t_0)}\, \mathrm{e}^{\mathrm{i}2\pi(t-t_0)\cdot\alpha} & \text{when } t > t_0, \\ 0 & \text{when } t \le t_0. \end{cases}$$

This is "vibration" at frequency $\alpha \in \mathbb{R}$, starting at time $t_0 \in \mathbb{R}$, decaying at rate $\varepsilon > 0$. Then

$$\widehat{s}(\nu) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu}\, s(t)\,\mathrm{d}t = \int_{t_0}^{\infty} \cdots\ \mathrm{d}t = \ldots = c\ \frac{\mathrm{e}^{-\mathrm{i}2\pi t_0\cdot\nu}}{1 + \mathrm{i}(\nu-\alpha)/\varepsilon}.$$

Energy densities $|s|^2$ in time and $|\widehat{s}|^2$ in frequency:

$$|s(t)|^2 = \begin{cases} (2\pi\varepsilon)^2\, \mathrm{e}^{-4\pi\varepsilon\,(t-t_0)} & \text{when } t > t_0, \\ 0 & \text{when } t \le t_0, \end{cases}$$

$$|\widehat{s}(\nu)|^2 = \frac{1}{1 + (\nu-\alpha)^2/\varepsilon^2} \qquad \text{for all } \nu \in \mathbb{R}.$$

Obviously, s cannot be retrieved back from $|s|^2$ and $|\widehat{s}|^2$, but we shall learn that $\widehat{s}$ contains essentially all the information about s: this enables useful operations on signals. Also, we shall learn that the energy is conserved in the Fourier transform: $\|\widehat{s}\|^2 = \|s\|^2$.

Schwartz test signals

Schwartz test signals $s : \mathbb{R} \to \mathbb{C}$ are "*smooth and rapidly decaying*" — let us be more precise:

Schwartz test signal $s \in \mathscr{S}(\mathbb{R})$ is an infinitely smooth function $s : \mathbb{R} \to \mathbb{C}$ for which

$$\lim_{|t| \to \infty} t^n \, s^{(m)}(t) = 0 \tag{4}$$

for all $m, n \in \mathbb{N} = \{0, 1, 2, 3, 4, 5, \cdots\}$. There are many test signals:

Example. If $s \in C^\infty(\mathbb{R})$ and $s(t) = 0$ whenever $|t| \geq 1$ then $s \in \mathscr{S}(\mathbb{R})$; e.g. define $s(t) := \exp(1/(t^2 - 1))$ for $|t| < 1$. Also Gaussian signals $t \mapsto \mathrm{e}^{at^2 + bt + c}$ are examples of Schwartz test functions (when $\mathrm{Re}(a) < 0$, $a, b, c \in \mathbb{C}$).

Example. Let $k \in \mathbb{N}$, $\lambda \in \mathbb{C}$, $r, s \in \mathscr{S}(\mathbb{R})$, and let $q : \mathbb{R} \to \mathbb{C}$ be a polynomial. Then $\lambda s, r + s, s^{(k)}, qs, rs \in \mathscr{S}(\mathbb{R})$.

Derivation and Fourier: If $s \in \mathscr{S}(\mathbb{R})$ then $\widehat{s} \in \mathscr{S}(\mathbb{R})$, because (writing $\widehat{r} = \widehat{t\,s}$ for $r(t) := t\,s(t)$ simply)

$$\widehat{s}\,'(\nu) \quad = \quad -\mathrm{i}2\pi\ \widehat{t\,s}\,(\nu), \tag{5}$$

$$\widehat{s'}\,(\nu) \quad = \quad +\mathrm{i}2\pi\,\nu\,\widehat{s}(\nu). \tag{6}$$

Formulas (5),(6) motivate the definition of Schwartz test signals! Hence the Fourier transform gives a linear mapping

$$(s \mapsto \widehat{s}) : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R}). \tag{7}$$

We leave checking (5) as an exercise, proving here only (6):

$$\begin{aligned}
\widehat{s'}\,(\nu) \quad &= \quad \int_{\mathbb{R}} s'(t)\,\mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu}\ \mathrm{d}t \\
&\overset{\textit{integrate by parts}}{=} \quad -\int_{\mathbb{R}} s(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu}\ \mathrm{d}t \\
&= \quad -\int_{\mathbb{R}} s(t)\,\mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu}\,(-\mathrm{i}2\pi\,\nu)\ \mathrm{d}t \\
&= \quad +\mathrm{i}2\pi\,\nu\ \widehat{s}(\nu).
\end{aligned}$$

Example. Let $s(t) = s_\varepsilon(t) = \mathrm{e}^{-\varepsilon\pi t^2}$, where $\varepsilon > 0$. First,

$$\begin{aligned} s'(t) &= -2\varepsilon\pi\, t\, s(t) \\ \overset{(6),(5)}{\Longrightarrow} \quad +\mathrm{i}2\pi\,\nu\,\widehat{s}(\nu) &= (\varepsilon/\mathrm{i})\,\widehat{s}\,'(\nu) \\ \Longleftrightarrow \quad \widehat{s}\,'(\nu) &= -(2\pi/\varepsilon)\,\nu\,\widehat{s}(\nu) \\ \overset{\text{Exercise!}}{\Longleftrightarrow} \quad \widehat{s}(\nu) &= \widehat{s}(0)\,\mathrm{e}^{-\pi\nu^2/\varepsilon}, \end{aligned}$$

and here

$$\begin{aligned} \widehat{s}(0) = \int_{\mathbb{R}} s(t)\;\mathrm{d}t &= \left[\int_{\mathbb{R}}\int_{\mathbb{R}} s(t)\,s(u)\;\mathrm{d}t\;\mathrm{d}u\right]^{1/2} \\ &= \left[\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{e}^{-\varepsilon\pi(t^2+u^2)}\;\mathrm{d}t\;\mathrm{d}u\right]^{1/2} \\ &\overset{\text{polar coordinates}}{=} \left[\int_0^\infty\int_0^{2\pi} \mathrm{e}^{-\varepsilon r^2}\; r\mathrm{d}\theta\;\mathrm{d}r\right]^{1/2} \\ &= \left[\int_0^\infty 2\pi r\,\mathrm{e}^{-\varepsilon\pi r^2}\;\mathrm{d}r\right]^{1/2} = \frac{1}{\sqrt{\varepsilon}}. \end{aligned}$$

Easy to remember: $\widehat{s} = s$, when $s(t) = \mathrm{e}^{-\pi t^2}$.

Fourier inverse transform!

For any $s \in \mathscr{S}(\mathbb{R})$, we find

$$\begin{aligned}
s(t) &= \lim_{0<\varepsilon\to 0} \int_{\mathbb{R}} s(u)\, \frac{1}{\sqrt{\varepsilon}}\, \mathrm{e}^{-\pi(u-t)^2/\varepsilon}\, \mathrm{d}u \\
&\overset{\text{previous example}}{=} \lim_{0<\varepsilon\to 0} \int_{\mathbb{R}} s(u) \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi(u-t)\cdot\nu}\, \mathrm{e}^{-\varepsilon\pi\nu^2}\, \mathrm{d}\nu\, \mathrm{d}u \\
&= \lim_{0<\varepsilon\to 0} \int_{\mathbb{R}} \mathrm{e}^{-\varepsilon\pi\nu^2}\, \mathrm{e}^{+\mathrm{i}2\pi t\cdot\nu} \int_{\mathbb{R}} s(u)\, \mathrm{e}^{-\mathrm{i}2\pi u\cdot\nu}\, \mathrm{d}u\, \mathrm{d}\nu \\
&= \int_{\mathbb{R}} \mathrm{e}^{+\mathrm{i}2\pi t\cdot\nu}\, \widehat{s}(\nu)\, \mathrm{d}\nu.
\end{aligned}$$

Remark! We just proved the **Fourier inverse formula**

$$s(t) = \int_{\mathbb{R}} \mathrm{e}^{+\mathrm{i}2\pi t\cdot\nu}\, \widehat{s}(\nu)\, \mathrm{d}\nu. \tag{8}$$

Thus the Fourier transform $(s \mapsto \widehat{s}) : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$ is bijective!

Notice that $\widehat{\widehat{s}}(t) = s(-t)$, and that $\widehat{\widehat{\widehat{\widehat{s}}}} = s$.

Fourier transform preserves energy

Inner product $\langle r, s\rangle := \int_{\mathbb{R}} r(t)\,\overline{s(t)}\,\mathrm{d}t \in \mathbb{C}$ between signals $r, s \in \mathscr{S}(\mathbb{R})$ is preserved by Fourier transform, because

$$\begin{aligned}
\langle \widehat{r}, \widehat{s}\rangle &= \int_{\mathbb{R}} \widehat{r}(\nu)\,\overline{\widehat{s}(\nu)}\,\mathrm{d}\nu \\
&= \int_{\mathbb{R}} \widehat{r}(\nu)\,\overline{\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu}\, s(t)\,\mathrm{d}t}\,\mathrm{d}\nu \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{+\mathrm{i}2\pi t\cdot\nu}\,\widehat{r}(\nu)\,\mathrm{d}\nu\,\overline{s(t)}\,\mathrm{d}t \\
&= \int_{\mathbb{R}} r(t)\,\overline{s(t)}\,\mathrm{d}t \quad = \quad \langle r, s\rangle.
\end{aligned}$$

Putting $r = s$, we see that Fourier transform preserves total **energy** $\|s\|^2 := \langle s, s\rangle = \int_{\mathbb{R}} |s(t)|^2\,\mathrm{d}t$ of signal $s \in \mathscr{S}(\mathbb{R})$:

$$\|\widehat{s}\|^2 = \|s\|^2. \tag{9}$$

Interpreting Fourier transform

We can think that the Fourier inverse formula

$$s(t) = \int_{\mathbb{R}} \widehat{s}(\nu)\, \mathrm{e}^{\mathrm{i}2\pi t\cdot\nu}\, \mathrm{d}\nu$$

describes signal s as an "infinite linear combination" of simple waves

$$t \mapsto \mathrm{e}^{\mathrm{i}2\pi t\cdot\nu} = \cos(2\pi t\cdot\nu) + \mathrm{i}\sin(2\pi t\cdot\nu).$$

Such a wave has frequency $\nu \in \mathbb{R}$ (which can also be negative!), and this wave has "weight" $\widehat{s}(\nu) \in \mathbb{C}$.

During the time interval $[a, b] \subset \mathbb{R}$ the signal s has energy

$$\int_a^b |s(t)|^2\, \mathrm{d}t,$$

and within the frequency band $[\alpha, \beta] \subset \mathbb{R}$ the amount of energy is

$$\int_\alpha^\beta |\widehat{s}(\nu)|^2\, \mathrm{d}\nu.$$

Convolution

Convolution of $r, s \in L^1(\mathbb{R})$ is signal $r * s : \mathbb{R} \to \mathbb{C}$ such that

$$r * s(t) = (r * s)(t) := \int_{\mathbb{R}} r(t-u)\, s(u)\, \mathrm{d}u. \tag{10}$$

The reader may then verify the absolute integrability of $r * s$:

$$\|r * s\|_{L^1} \leq \|r\|_{L^1}\, \|s\|_{L^1} < \infty.$$

"Convolution in time" is "multiplication in frequency":

$$\widehat{r * s} = \widehat{r}\,\widehat{s}. \tag{11}$$

This is useful in signal processing! Moreover,

$$(r * s)' = r' * s, \tag{12}$$

if also r' is absolutely integrable: hence convolution makes signal s smoother! Furthermore, $r * s \in \mathscr{S}(\mathbb{R})$ when $r, s \in \mathscr{S}(\mathbb{R})$.

Symmetries of time and frequency

Time translation of signal $s \in \mathscr{S}(\mathbb{R})$ by time-lag $b \in \mathbb{R}$ is signal $T_b s \in \mathscr{S}(\mathbb{R})$, where

$$T_b s(t) := s(t - b). \tag{13}$$

Frequency modulation of $s \in \mathscr{S}(\mathbb{R})$ by frequency-lag $\alpha \in \mathbb{R}$ is signal $M_\alpha s \in \mathscr{S}(\mathbb{R})$, where

$$M_\alpha s(t) := \mathrm{e}^{+\mathrm{i}2\pi t\cdot\alpha} s(t). \tag{14}$$

After Fourier transforms: $\widehat{M_\alpha s} = T_\alpha \widehat{s}$ and $\widehat{T_b s} = M_{-b}\widehat{s}$, that is

$$\begin{aligned} \widehat{M_\alpha s}(\nu) &= T_\alpha \widehat{s}(\nu), \\ \widehat{T_b s}(\nu) &= M_{-b}\widehat{s}(\nu). \end{aligned}$$

Integral operators

We want to transform input signal $s \in \mathscr{S}(\mathbb{R})$
to output signal $As = A(s) \in \mathscr{S}(\mathbb{R})$.
Suppose A is *linear*, i.e.

$$\begin{aligned} A(r+s) &= A(r) + A(s) \quad \text{and} \\ A(\lambda s) &= \lambda\, A(s) \end{aligned}$$

for all signals $r, s \in \mathscr{S}(\mathbb{R})$ and constants $\lambda \in \mathbb{C}$.
Linear transform A presented as an **integral operator**:

$$As(t) = \int_{\mathbb{R}} K_A(t,u)\, s(u)\, \mathrm{d}u, \tag{15}$$

where K_A is the **kernel** of A.
Remark: integral operator A has "essentially unique" kernel K_A (provided that $s \mapsto As$ is "naturally continuous" — precise statement in so-called *Schwartz kernels theorem*).

Time-invariant operators

Let operator A be **time-invariant**: $T_b A = A T_b$ for all $b \in \mathbb{R}$, i.e.

$$T_b A s(t) = A T_b s(t) \tag{16}$$

for all signals $s : \mathbb{R} \to \mathbb{C}$ and for all $t, b \in \mathbb{R}$;
in other words, $A = T_{-b} A T_b$, which means

$$\begin{aligned}
\int_{\mathbb{R}} K_A(t,u)\ s(u)\ \mathrm{d}u &= As(t) \ = \ T_{-b} A T_b s(t) \ = \ A T_b s(t+b) \\
&= \int_{\mathbb{R}} K_A(t+b,u)\ T_b s(u)\ \mathrm{d}u \\
&= \int_{\mathbb{R}} K_A(t+b,u)\ s(u-b)\ \mathrm{d}u \\
&= \int_{\mathbb{R}} K_A(t+b,u+b)\ s(u)\ \mathrm{d}u.
\end{aligned}$$

Thus $K_A(t,u) = K_A(t+b,u+b)$ for all $b,t,u \in \mathbb{R}$, especially
$K_A(t,u) = K_A(t-u,0) = r(t-u)$ for some signal $r : \mathbb{R} \to \mathbb{C}$...
... Hey, $As = r * s$ is a convolution!!!

Extending Fourier analysis

Test signals $s \in \mathscr{S}(\mathbb{R})$ are "tame"; we shall extend Fourier analysis to "wilder" signals! "Size" of signal $s : \mathbb{R} \to \mathbb{C}$ is measured by norms

$$\|s\|_{L^p} \overset{1\le p<\infty}{:=} \left[\int_{\mathbb{R}} |s(t)|^p \,\mathrm{d}t\right]^{1/p},$$

$$\|s\|_{L^\infty} := \operatorname{ess\,sup}_{t\in\mathbb{R}}|s(t)| \overset{\text{If } s \text{ continuous}}{=} \sup_{t\in\mathbb{R}} |s(t)| = \lim_{p\to\infty} \|s\|_{L^p}.$$

We denote $s \in L^p(\mathbb{R})$, if $\|s\|_{L^p} < \infty$.
Spaces $L^p(\mathbb{R})$ are so-called **Lebesgue spaces**.

$s \in L^1(\mathbb{R})$ is *absolutely integrable*: $\int_{\mathbb{R}} |s(t)| \,\mathrm{d}t = \|s\|_{L^1}$.
$s \in L^2(\mathbb{R})$ has *finite energy*: $\|s\|^2 = \int_{\mathbb{R}} |s(t)|^2 \,\mathrm{d}t = [\|s\|_{L^2}]^2$.
$s \in L^\infty(\mathbb{R})$ is *essentially bounded*: $|s(t)| \le \|s\|_{L^\infty}$ for almost all t.

Write $r = s$ if $\|r - s\|_{L^p} = 0$ for $r, s \in L^p(\mathbb{R})$
(which happens if $r(t) = s(t)$ for almost every $t \in \mathbb{R}$).

Here $\mathscr{S}(\mathbb{R}) \subset L^p(\mathbb{R})$ for all $p \in [1, \infty]$. Functions $s \in L^p(\mathbb{R})$ certainly can be discontinuous. Nevertheless, if $s \in L^1(\mathbb{R})$ and

$$r(t) := \int_0^t s(u)\,\mathrm{d}u$$

then $r \in L^\infty(\mathbb{R})$ (satisfying $\|r\|_{L^\infty} \leq \|s\|_{L^1}$ clearly),
and $r' = s$ in sense that $r'(t) = s(t)$ for almost all $t \in \mathbb{R}$
(this is so-called **Lebesgue differentiation theorem**).

If $1 < p < \infty$ and $s \in L^p(\mathbb{R})$ then $s = s_1 + s_\infty$,
where $s_1 \in L^1(\mathbb{R})$ and $s_\infty \in L^\infty(\mathbb{R})$. Why? Simply define

$$s_\infty(t) := \begin{cases} s(t) & \text{when } |s(t)| \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, if we want to find Fourier transform $\widehat{s}$ for signal $s \in L^p(\mathbb{R})$,
we need to understand the special cases $p = 1$ and $p = \infty$.
For $p = 1$, we already have the nice Fourier integrals.
Case $p = \infty$ leads naturally to so-called *distributions*
(which are a generalization of ordinary functions).

Density of $\mathscr{S}(\mathbb{R})$ in $L^p(\mathbb{R})$, when $1 \leq p < \infty$

Let us try to approximate $s \in L^p(\mathbb{R})$ by test functions $s_k \in \mathscr{S}(\mathbb{R})$. If $g, r \in \mathscr{S}(\mathbb{R})$, then $g * (rs) \in \mathscr{S}(\mathbb{R})$: smoothing by convolution! For $k \in \mathbb{Z}^+$, define $g_k, r_k \in \mathscr{S}(\mathbb{R})$ by

$$r_k(\nu) = \widehat{g_k}(\nu) := \mathrm{e}^{-\pi(\nu/k)^2},$$

so that $s_k := g_k * (r_k\, s) \in \mathscr{S}(\mathbb{R})$. Now, if $1 \leq p < \infty$ then

$$\lim_{k\to\infty} \|s - s_k\|_{L^p} = 0, \quad \text{in other words} \quad s = \lim_{k\to\infty} s_k \quad \text{in} \quad L^p(\mathbb{R}).$$

This means that $\mathscr{S}(\mathbb{R})$ is **dense** in $L^p(\mathbb{R})$, when $1 \leq p < \infty$.

$\mathscr{S}(\mathbb{R})$ is not dense in $L^\infty(\mathbb{R})$: for instance, think of the constant function $1 \in L^\infty(\mathbb{R})$, for which $\|s - 1\|_{L^\infty} \geq 1$ for every $s \in \mathscr{S}(\mathbb{R})$. Thereby we cannot define Fourier transform for $s \in L^\infty(\mathbb{R})$ by bounded linear extension of $(s \mapsto \widehat{s}) : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R})$. However, there is another method, which we later shall learn.

Extending Fourier transform to $L^2(\mathbb{R})$

We have the linear energy-preserving Fourier integral transform

$$(s \mapsto \widehat{s}) : \mathscr{S}(\mathbb{R}) \to \mathscr{S}(\mathbb{R}), \quad \|\widehat{s}\|^2 = \|s\|^2.$$

If $s \in L^2(\mathbb{R})$, by density of $\mathscr{S}(\mathbb{R}) \subset L^2(\mathbb{R})$, take $s_k \in \mathscr{S}(\mathbb{R})$ so that

$$\lim_{k\to\infty} \|s - s_k\| = 0, \quad \text{i.e.} \quad s = \lim_{k\to\infty} s_k \quad \text{in} \quad L^2(\mathbb{R}).$$

No matter which approximations s_k we choose,
the energy-preservation guarantees the uniqueness of the limit

$$\widehat{s} := \lim_{k\to\infty} \widehat{s_k} \quad \in \quad L^2(\mathbb{R}).$$

This defines the linear energy-preserving Fourier transform

$$(s \mapsto \widehat{s}) : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad \|\widehat{s}\|^2 = \|s\|^2. \tag{17}$$

This is automatically a bijection, and also *unitary*, which means $\langle \widehat{r}, \widehat{s} \rangle = \langle r, s \rangle$ for all $r, s \in L^2(\mathbb{R})$, where the inner product is

$$\langle r, s \rangle := \int_{\mathbb{R}} r(t)\, s(t)^* \,\mathrm{d}t.$$

Integrals $\widehat{s}(\nu) := \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu}\, s(t)\,\mathrm{d}t$ define the Fourier transform for $s \in L^1(\mathbb{R})$. However, such integrals do not converge absolutely if $s \notin L^1(\mathbb{R})$. For $s \in L^2(\mathbb{R})$ and $\psi \in \mathscr{S}(\mathbb{R})$ we have

$$\begin{aligned} \langle \widehat{\widehat{s}}, \widehat{\widehat{\psi}} \rangle &= \langle \widehat{s}, \widehat{\psi} \rangle = \langle s, \psi \rangle \\ &= \int_{\mathbb{R}} s(t)\psi(t)^* \mathrm{d}t = \int_{\mathbb{R}} s(t)\widehat{\widehat{\psi}}(-t)^* \mathrm{d}t = \int_{\mathbb{R}} s(-t)\widehat{\widehat{\psi}}(t)^* \mathrm{d}t : \end{aligned}$$

thus $\widehat{\widehat{s}}(t) = s(-t)$ for almost every $t \in \mathbb{R}$.

Example. Let $s(t) = 1$ for $|t| < 1/2$, and $s(t) = 0$ otherwise. Then $s \in L^1(\mathbb{R})$, and $\widehat{s} = \mathrm{sinc} \in L^\infty(\mathbb{R})$ is the *cardinal sine*, where

$$\mathrm{sinc}(\nu) := \begin{cases} \frac{\sin(\pi\nu)}{\pi\nu} & \text{for } \nu \neq 0, \\ 1 & \text{for } \nu = 0. \end{cases}$$

Now $\mathrm{sinc} \in L^2(\mathbb{R})$ but $\mathrm{sinc} \notin L^1(\mathbb{R})$. However, $\widehat{\mathrm{sinc}}(t) = \widehat{\widehat{s}}(t) = s(-t) = s(+t)$ for almost every $t \in \mathbb{R}$, so that $\widehat{\mathrm{sinc}} = s \in L^2(\mathbb{R})$.

Extending Fourier transform beyond L^p-spaces

For $s \in L^\infty(\mathbb{R})$, $m \in \mathbb{N}$, polynomial $r : \mathbb{R} \to \mathbb{C}$ and $\psi \in \mathscr{S}(\mathbb{R})$, let

$$\langle r\, s^{(m)}, \psi \rangle := (-1)^m \int_{\mathbb{R}} s(t)\, (\psi^*\, r)^{(m)}(t)\, \mathrm{d}t \tag{18}$$

(where the m^{th} derivative $s^{(m)}$ makes classically sense if $s \in \mathscr{S}(\mathbb{R})$ — formula (18) is just inspired from formal integration by parts).
Here $s^{(m)}$ is called the m^{th} *distribution derivative* of $s \in L^\infty(\mathbb{R})$.
If $r_1, \cdots, r_n : \mathbb{R} \to \mathbb{C}$ are polynomials and $s_1, \cdots, s_n \in L^\infty(\mathbb{R})$, then

$$s = \sum_{m=0}^{n} r_m\, s_m^{(m)} \tag{19}$$

is called a **Schwartz tempered distribution** $s \in \mathscr{S}'(\mathbb{R})$.
The Fourier transform $\widehat{s} \in \mathscr{S}'(\mathbb{R})$ is then defined by

$$\langle \widehat{s}, \widehat{\psi} \rangle := \langle s, \psi \rangle = \sum_{m=1}^{n} \langle r_m\, s_m^{(m)}, \psi \rangle \tag{20}$$

(which is again classically justified if $s \in \mathscr{S}(\mathbb{R})$).

Now we have obtained the bijective Fourier transform

$$(s \mapsto \widehat{s}) : \mathscr{S}'(\mathbb{R}) \to \mathscr{S}'(\mathbb{R}). \tag{21}$$

The space $\mathscr{S}'(\mathbb{R})$ of tempered distributions is rather large:

Example. $L^p(\mathbb{R}) \subset \mathscr{S}'(\mathbb{R})$ for every $p \in [1, \infty]$. If $s \in \mathscr{S}'(\mathbb{R})$ then the distribution derivatives $s^{(m)} \in \mathscr{S}'(\mathbb{R})$ for every $m \in \mathbb{N}$.

Example. Let $s(t) = e_\beta(t) := \mathrm{e}^{\mathrm{i}2\pi t \cdot \beta}$. Then $e_\beta \in L^\infty(\mathbb{R})$, and

$$\langle \widehat{e_\beta}, \widehat{\psi} \rangle := \langle e_\beta, \psi \rangle = \int_{\mathbb{R}} \mathrm{e}^{+\mathrm{i}2\pi t \cdot \beta} \psi(t)^* \mathrm{d}t = \widehat{\psi}(\beta)^* =: \int_{\mathbb{R}} \delta_\beta(\nu) \widehat{\psi}(\nu)^* \mathrm{d}\nu,$$

where $\delta_\beta := \widehat{e_\beta} \notin L^p(\mathbb{R})$ is the **Dirac delta distribution** at $\beta \in \mathbb{R}$.

Think $\delta_b \in \mathscr{S}'(\mathbb{R})$ as a unit mass (or a unit impulse) at $t = b$. Roughly, $\delta_b(t) = 0$ if $t \neq b$, but beware: δ_b is not a function, because if s is a function such that $s(t) = 0$ for almost every $t \in \mathbb{R}$ then $\int_{\mathbb{R}} s(t)\, \psi(t)^* \,\mathrm{d}t = 0$ for all $\psi \in \mathscr{S}(\mathbb{R})$. No function $s : \mathbb{R} \to \mathbb{C}$ satisfies $\int_{\mathbb{R}} s(t)\, \psi(t)^* \,\mathrm{d}t = \psi(b)^*$ for all $\psi \in \mathscr{S}(\mathbb{R})$.

Example. Dirac delta $\delta_b \notin L^p(\mathbb{R})$ for any $p \in [1, \infty]$. Yet here

$$\langle \widehat{\delta_b}, \widehat{\psi} \rangle := \langle \delta_b, \psi \rangle = \psi(b)^* = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi b\cdot\nu}\, \widehat{\psi}(\nu)^*\, \mathrm{d}\nu = \langle e_{-b}, \widehat{\psi} \rangle,$$

giving $\widehat{\delta_b} = e_{-b} \in L^\infty(\mathbb{R})$. An alternative, informal computation is

$$\widehat{\delta_b}(\nu) = \int_{\mathbb{R}} \delta_b(t)\, \mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu}\, \mathrm{d}t = \mathrm{e}^{-\mathrm{i}2\pi b\cdot\nu} = e_{-b}(\nu).$$

Example. *Signum* function $\mathrm{sgn} \in L^\infty(\mathbb{R})$ is defined by $\mathrm{sgn}(0) = 0$ and $\mathrm{sgn}(t) := t/|t|$ otherwise. Notice that the derivative

$$\mathrm{sgn}'(t) := \lim_{h\to 0} \frac{\mathrm{sgn}(t+h) - \mathrm{sgn}(t)}{h} \in \mathbb{R}$$

exists if and only if $t \neq 0$. For distribution derivative $\mathrm{sgn}' = \mathrm{sgn}^{(1)}$,

$$\begin{aligned} \langle \mathrm{sgn}', \psi \rangle &:= -\langle \mathrm{sgn}, \psi' \rangle = -\int_{\mathbb{R}} \mathrm{sgn}(t)\, \psi'(t)^*\, \mathrm{d}t \\ &= \int_{-\infty}^{0} \psi'(t)^*\, \mathrm{d}t - \int_{0}^{\infty} \psi'(t)^*\, \mathrm{d}t = \psi(0)^* + \psi(0)^*. \end{aligned}$$

Hence the distribution derivative is $\mathrm{sgn}' = 2\,\delta_0 \in \mathscr{S}'(\mathbb{R})$.

Example. For $\varepsilon > 0$, let us define $s_\varepsilon \in L^1(\mathbb{R})$ by $s_\varepsilon(t) := \mathrm{e}^{-\varepsilon|t|}\operatorname{sgn}(t)$. Then $\|s_\varepsilon - \operatorname{sgn}\|_{L^\infty} \not\to 0$ as $\varepsilon \to 0^+$, yet

$$\lim_{\varepsilon\to 0^+} s_\varepsilon(t) = \operatorname{sgn}(t),$$

and for $\nu \neq 0$ we have

$$\widehat{sgn}(\nu) = \lim_{\varepsilon\to 0^+} \widehat{s_\varepsilon}(\nu) = \ldots = \frac{1}{\mathrm{i}\pi\nu}.$$

Thus if $r(u) := \frac{1}{\pi u}$ for $u \neq 0$, then $\widehat{r}(\nu) = +\mathrm{i}\operatorname{sgn}(-\nu) = -\mathrm{i}\operatorname{sgn}(\nu)$ formally. This suggests that the *Hilbert transform* $H = (s \mapsto Hs) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, for which

$$\widehat{Hs}(\nu) = -\mathrm{i}\operatorname{sgn}(\nu)\,\widehat{s}(\nu),$$

should satisfy convolution-type singular integral formula

$$Hs(t) = \int_{\mathbb{R}} \frac{s(t-u)}{\pi u}\mathrm{d}u := \lim_{\varepsilon\to 0^+}\left(\int_{-\infty}^{-\varepsilon} \frac{s(t-u)}{\pi u}\mathrm{d}u + \int_{\varepsilon}^{\infty} \frac{s(t-u)}{\pi u}\mathrm{d}u\right).$$

Friendly interpretation of Fourier integrals

For the absolute convergence of Fourier integrals

$$s(t) = \int_{\mathbb{R}} \mathrm{e}^{+\mathrm{i}2\pi t\cdot\nu}\, \widehat{s}(\nu)\, \mathrm{d}\nu = \int_{\mathbb{R}} \mathrm{e}^{+\mathrm{i}2\pi t\cdot\nu} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi u\cdot\nu}\, s(u)\, \mathrm{d}u\, \mathrm{d}\nu,$$

we must have $s, \widehat{s} \in L^1(\mathbb{R})$, and then $s, \widehat{s}$ are also continuous and belong to all L^p-spaces: this is true certainly if $s, s', s'' \in L^1(\mathbb{R})$ (or more generally if $s, s' \in L^1(\mathbb{R})$ and $s' \in L^2(\mathbb{R})$).

However, we have been able to extend our Fourier interpretations beyound L^p-spaces to tempered distributions. Thus, it is not harmful to write such Fourier integral formulas for signals outside $L^1(\mathbb{R})$, too! For $s \in \mathscr{S}'(\mathbb{R})$, in sense of distributions,

$$\int_{\mathbb{R}} \mathrm{e}^{+\mathrm{i}2\pi t\cdot\nu}\, \widehat{s}(\nu)\, \mathrm{d}\nu \;=\; \int_{\mathbb{R}} \mathrm{e}^{+\mathrm{i}2\pi t\cdot\nu} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi u\cdot\nu}\, s(u)\, \mathrm{d}u\, \mathrm{d}\nu$$

$$=\; \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi(t-u)\cdot\nu}\, \mathrm{d}\nu\, s(u)\, \mathrm{d}u \;=\; \int_{\mathbb{R}} \delta_0(t-u)\, s(u)\, \mathrm{d}u \;=\; s(t).$$

Fourier bijections

So, we have the bijective time-to-frequency Fourier transforms

$$\begin{array}{ccccc} \mathscr{S}(\mathbb{R}) & \subset & L^2(\mathbb{R}) & \subset & \mathscr{S}'(\mathbb{R}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathscr{S}(\mathbb{R}) & \subset & L^2(\mathbb{R}) & \subset & \mathscr{S}'(\mathbb{R}). \end{array}$$

$\mathscr{S}(\mathbb{R})$ contains all the smooth rapidly decaying signals.
$L^2(\mathbb{R})$ contains all the finite energy signals.
$\mathscr{S}'(\mathbb{R})$ contains "nearly all the signals we ever meet".

With these Fourier bijections, we may present the signal
either in time **or** in frequency,
whatever is convenient for manipulation.

At the end of this course, we shall learn how to operate
both in time **and** in frequency, simultaneously!

Fourier integral in dimension $d \in \mathbb{Z}^+ = \{1, 2, 3, 4, 5, \cdots\}$

Fourier transform $\widehat{s} : \mathbb{R}^d \to \mathbb{C}$ for signal $s : \mathbb{R}^d \to \mathbb{C}$ is given by

$$\widehat{s}(\nu) := \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu}\ s(t)\ \mathrm{d}t, \tag{22}$$

where $t = (t_1, \cdots, t_d) \in \mathbb{R}^d$, $\nu = (\nu_1, \cdots, \nu_d) \in \mathbb{R}^d$,
$t \cdot \nu = \sum_{k=1}^{d} t_k \cdot \nu_k = t_1\,\nu_1 + \cdots + t_d\,\nu_d \in \mathbb{R}$,

$$\int_{\mathbb{R}^d} \cdots\ \mathrm{d}t = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \cdots\ \mathrm{d}t_1\ \cdots\ \mathrm{d}t_d.$$

Energy $\|s\|^2 := \int_{\mathbb{R}^d} |s(t)|^2\,\mathrm{d}t$, and for example

$$\begin{aligned} s(t) &= \int_{\mathbb{R}^d} \mathrm{e}^{+\mathrm{i}2\pi t\cdot\nu}\ \widehat{s}(\nu)\ \mathrm{d}\nu, \\ \|s\|^2 &= \|\widehat{s}\|^2, \\ r * s(t) &:= \int_{\mathbb{R}^d} r(t-u)\ s(u)\ \mathrm{d}u, \\ \widehat{r * s}(\nu) &= \widehat{r}(\nu)\ \widehat{s}(\nu). \end{aligned}$$

Analog periodic world (A1)

Signal $s : \mathbb{R} \to \mathbb{C}$ is P-**periodic** if $T_P s = s$, meaning $s(t - P) = s(t)$ for all $t \in \mathbb{R}$: in this case, we denote $s : \mathbb{R}/P\mathbb{Z} \to \mathbb{C}$. Without losing generality, we deal with 1-periodic signals

$$s : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$$

for which $s(t-1) = s(t)$ for all $t \in \mathbb{R}$; then the **Fourier coefficient transform** $\mathcal{F}_{\mathbb{R}/\mathbb{Z}} s = \widehat{s} : \mathbb{Z} \to \mathbb{C}$ is defined by

$$\widehat{s}(\nu) := \int_{\mathbb{R}/\mathbb{Z}} \mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu}\, s(t)\, \mathrm{d}t = \int_0^1 \mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu}\, s(t)\, \mathrm{d}t. \qquad (23)$$

Exercise: show that $\widehat{s}(\nu) = c_\nu \in \mathbb{C}$, when $s : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ is given by

$$s(t) := \sum_{k\in\mathbb{Z}} c_k\, \mathrm{e}^{\mathrm{i}2\pi t\cdot k} = \sum_{k=-\infty}^{\infty} c_k\, \mathrm{e}^{\mathrm{i}2\pi t\cdot k}$$

(naturally, provided that signal s is “nice enough”).

For periodic signal $s : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$, Fourier coefficients

$$\widehat{s}(\nu) = \int_{\mathbb{R}/\mathbb{Z}} \mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu} s(t)\, \mathrm{d}t \quad \in \quad \mathbb{C}.$$

It turns out that "nice enough" $s : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ can be recovered from its Fourier coefficients by **Fourier series**

$$s(t) = \sum_{\nu\in\mathbb{Z}} \mathrm{e}^{+\mathrm{i}2\pi t\cdot\nu}\, \widehat{s}(\nu) = \sum_{\nu=-\infty}^{+\infty} \mathrm{e}^{+\mathrm{i}2\pi t\cdot\nu}\, \widehat{s}(\nu). \qquad (24)$$

Thus **periodic analog signal** $s : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ has the same information content as **non-periodic digital signal** $\widehat{s} : \mathbb{Z} \to \mathbb{C}$; using the signal classification presented in the beginning of the course, this means that classes (A1) and (D0) are dual to each other by Fourier transform, so that properties in (A1) have corresponding "mirrored" properties in (D0), and vice versa.

(A1) Where do Fourier series come from?

Poisson kernel φ_r, for which $0 < \varphi_r(t) < \infty$ and $\int_0^1 \varphi_r(t)\,\mathrm{d}t = 1$:

$$\varphi_r(t) := \sum_{\nu \in \mathbb{Z}} r^{|\nu|}\,\mathrm{e}^{\mathrm{i}2\pi t\cdot\nu} = \frac{1 - r^2}{1 + r^2 - 2r\cos(2\pi t)}, \qquad (25)$$

where $0 < r < 1$. Then for smooth $s : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ we have

$$\begin{aligned} s(t) &= \lim_{r\to 1^-} \int_0^1 s(u)\,\varphi_r(t-u)\,\mathrm{d}u \\ &= \lim_{r\to 1^-} \int_0^1 s(u) \sum_{\nu\in\mathbb{Z}} r^{|\nu|}\,\mathrm{e}^{\mathrm{i}2\pi(t-u)\cdot\nu}\,\mathrm{d}u \\ &= \lim_{r\to 1^-} \sum_{\nu\in\mathbb{Z}} \widehat{s}(\nu)\,r^{|\nu|}\,\mathrm{e}^{\mathrm{i}2\pi t\cdot\nu} \\ &= \sum_{\nu\in\mathbb{Z}} \widehat{s}(\nu)\,\mathrm{e}^{\mathrm{i}2\pi t\cdot\nu}. \end{aligned}$$

Energy conservation in Fourier coefficients and series

Let $r, s : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$, so that $\widehat{r}, \widehat{s} : \mathbb{Z} \to \mathbb{C}$. Then

$$
\begin{aligned}
\langle \widehat{r}, \widehat{s} \rangle \quad &:= \quad \sum_{\nu \in \mathbb{Z}} \widehat{r}(\nu)\, \overline{\widehat{s}(\nu)} \\
&= \quad \sum_{\nu \in \mathbb{Z}} \widehat{r}(\nu) \overline{\int_{\mathbb{R}/\mathbb{Z}} \mathrm{e}^{-\mathrm{i}2\pi t \cdot \nu}\, s(t)\, \mathrm{d}t} \\
&= \quad \int_{\mathbb{R}/\mathbb{Z}} \sum_{\nu \in \mathbb{Z}} \mathrm{e}^{+\mathrm{i}2\pi t \cdot \nu}\, \widehat{r}(\nu)\, \overline{s(t)}\, \mathrm{d}t \\
&= \quad \int_{\mathbb{R}/\mathbb{Z}} r(t)\, \overline{s(t)}\, \mathrm{d}t \quad =: \quad \langle r, s \rangle.
\end{aligned}
$$

We see that Fourier coefficient/series transform preserves energy

$$
\|s\|^2 := \langle s, s \rangle = \langle \widehat{s}, \widehat{s} \rangle =: \|\widehat{s}\|^2. \tag{26}
$$

(A1) Convolution of periodic signals

Convolution $r * s : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ of periodic signals $r, s : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ is defined by

$$r * s(t) := \int_{\mathbb{R}/\mathbb{Z}} r(t-u)\, s(u)\, \mathrm{d}u. \tag{27}$$

By easy computation, we see that $\widehat{r * s} = \widehat{r}\,\widehat{s}$:

$$\widehat{r * s}(\nu) = \widehat{r}(\nu)\, \widehat{s}(\nu).$$

Naturally, periodic convolution has smoothing properties:

$$(r * s)'(t) = r' * s(t).$$

Thus, convolution works in similar manner for both periodic and non-periodic signals!

Periodization and Poisson summation formula

Periodization of signal $s : \mathbb{R} \to \mathbb{C}$ is $\mathcal{P}s : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$, where

$$\mathcal{P}s(t) := \sum_{k\in\mathbb{Z}} s(t-k).$$

Their Fourier transforms $\widehat{s} : \mathbb{R} \to \mathbb{C}$ and $\widehat{\mathcal{P}s} : \mathbb{Z} \to \mathbb{C}$ satisfy

$$\begin{aligned}
\widehat{\mathcal{P}s}(\nu) &= \int_0^1 \mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu} \sum_{k\in\mathbb{Z}} s(t-k)\,\mathrm{d}t \\
&= \sum_{k\in\mathbb{Z}} \int_0^1 \mathrm{e}^{-\mathrm{i}2\pi(t-k)\cdot\nu}\ s(t-k)\,\mathrm{d}t \\
&= \int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu}\ s(t)\,\mathrm{d}t \ = \ \widehat{s}(\nu).
\end{aligned}$$

Result $\widehat{\mathcal{P}s}(\nu) = \widehat{s}(\nu)$ together with $\sum_\nu \widehat{\mathcal{P}s}(\nu) = \mathcal{P}s(0)$ yields **Poisson summation formula**

$$\sum_{\nu\in\mathbb{Z}} \widehat{s}(\nu) \ = \ \sum_{k\in\mathbb{Z}} s(k). \tag{28}$$

Digital non-periodic world (D0), or DTFT

Fourier transform of digital signal $s : \mathbb{Z} \to \mathbb{C}$ is periodic signal $\mathcal{F}_{\mathbb{Z}}(s) = \widehat{s} : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ defined by

$$\widehat{s}(\nu) := \sum_{t \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}2\pi t \cdot \nu} \; s(t). \tag{29}$$

This is called **Discrete Time Fourier Transform** (DTFT).
Remark: this is essentially similar to the previous Fourier series case (apart from the sign of the imaginary unit i).
For digital signals $r, s : \mathbb{Z} \to \mathbb{C}$, we define the convolution $r * s : \mathbb{Z} \to \mathbb{C}$ by

$$r * s(t) := \sum_{u \in \mathbb{Z}} r(t - u) \; s(u). \tag{30}$$

The reader may check that $\widehat{r * s} = \widehat{r}\,\widehat{s}$, that is

$$\widehat{r * s}(\nu) = \widehat{r}(\nu) \; \widehat{s}(\nu).$$

Inverse transform to DTFT

For $s : \mathbb{Z} \to \mathbb{C}$ we have DTFT $\widehat{s} : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$, where

$$\widehat{s}(\nu) := \sum_{t \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}2\pi t \cdot \nu} \; s(t).$$

The inverse transform is verified by a direct calculation:

$$\begin{aligned} \int_{\mathbb{R}/\mathbb{Z}} \mathrm{e}^{+\mathrm{i}2\pi t \cdot \nu} \; \widehat{s}(\nu) \; \mathrm{d}\nu &= \int_{\mathbb{R}/\mathbb{Z}} \mathrm{e}^{+\mathrm{i}2\pi t \cdot \nu} \sum_{u \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}2\pi u \cdot \nu} s(u) \; \mathrm{d}\nu \\ &= \sum_{u \in \mathbb{Z}} s(u) \int_0^1 \mathrm{e}^{\mathrm{i}2\pi (t-u) \cdot \nu} \; \mathrm{d}\nu \\ &= s(t). \end{aligned}$$

Well, no wonder: this is just because signal classes (A0) and (D1) are dual to each other by Fourier transform! Thus, no need to check conservation of energy again.

From Poisson summation to sampling

Poisson summation formula $\sum_{\nu \in \mathbb{Z}} \widehat{s}(\nu) = \sum_{k \in \mathbb{Z}} s(k)$ is equivalent to

$$\sum_{\alpha \in \mathbb{Z}} \widehat{s}(\nu - \alpha) = \sum_{k \in \mathbb{Z}} s(k)\, \mathrm{e}^{-\mathrm{i}2\pi k \cdot \nu}. \tag{31}$$

Now suppose that $\widehat{s_1}(\nu) = 0$ whenever $|\nu| \geq 1/2$: then

$$\begin{aligned} \widehat{s_1}(\nu) &= \mathbf{1}_{]-1/2,+1/2[}(\nu) \sum_{\alpha \in \mathbb{Z}} \widehat{s_1}(\nu - \alpha) \\ &\overset{(31)}{=} \mathbf{1}_{]-1/2,+1/2[}(\nu) \sum_{k \in \mathbb{Z}} s_1(k)\, \mathrm{e}^{-\mathrm{i}2\pi k \cdot \nu} \\ &= \sum_{k \in \mathbb{Z}} s_1(k)\, \mathrm{e}^{-\mathrm{i}2\pi k \cdot \nu}\, \mathbf{1}_{]-1/2,+1/2[}(\nu), \end{aligned}$$

leading to **normalized Whittaker–Shannon sampling formula**

$$s_1(t) = \sum_{k \in \mathbb{Z}} s_1(k)\, \mathrm{sinc}(t - k). \tag{32}$$

... From this, we get **Whittaker–Shannon sampling formula**

$$s(t) = \sum_{k \in \mathbb{Z}} s(\frac{k}{2B}) \operatorname{sinc}(2Bt - k), \qquad (33)$$

which is valid if $\widehat{s}(\nu) = 0$ whenever $|\nu| \geq B$.
Related to this formula, **Nyquist–Shannon sampling theorem** says: If analog signal $s : \mathbb{R} \to \mathbb{C}$ is **band-limited** (meaning $\widehat{s}(\nu) = 0$ whenever $|\nu| \geq B$), then we are able to reconstruct it from its equispaced sampled values, i.e. from the corresponding digital signal $r : \mathbb{Z} \to \mathbb{C}$, where

$$r(k) := s(k/(2B)).$$

In other words, Whittaker–Shannon formula builds a bridge between non-periodic analog signals and non-periodic digital signals!

Digital periodic world (D1), or DFT

N-periodic digital signal $s : \mathbb{Z} \to \mathbb{C}$ satisfies $s(t - N) = s(t)$ for all $t \in \mathbb{Z}$: then we denote

$$s : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}. \tag{34}$$

Its **discrete Fourier transform** (DFT) $\widehat{s} : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ is defined by

$$\widehat{s}(\nu) := \sum_{t=1}^{N} \mathrm{e}^{-\mathrm{i}2\pi t \cdot \nu/N} \, s(t). \tag{35}$$

Notice that in the exponential we have $t \cdot \nu/N$ instead of $t \cdot \nu$.
Exercise: show that the inverse $\widehat{s} \mapsto s$ of DFT is given by

$$s(t) = \frac{1}{N} \sum_{\nu=1}^{N} \mathrm{e}^{+\mathrm{i}2\pi t \cdot \nu/N} \, \widehat{s}(\nu). \tag{36}$$

Notice the factor $\frac{1}{N}$ in this formula!

Energy and convolution

Exercise: defining here **energy** $\|s\|^2 := \sum_{t=1}^{N} |s(t)|^2$, find constant c_N such that for all signals $s : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$

$$\|s\|^2 = c_N \, \|\widehat{s}\|^2. \tag{37}$$

Hence "energy is conserved up to a constant".
For digital signals $r, s : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, we define the discrete **convolution** $r * s : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ by

$$r * s(t) := \sum_{u=1}^{N} r(t-u) \; s(u). \tag{38}$$

The reader may check that $\widehat{r * s} = \widehat{r}\,\widehat{s}$, that is

$$\widehat{r * s}(\nu) = \widehat{r}(\nu) \; \widehat{s}(\nu).$$

DFT related to DTFT (D1 vs. D0)

For "nice" non-periodic $s : \mathbb{Z} \to \mathbb{C}$, define $s_N : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ by

$$s_N(t) := \sum_{k\in\mathbb{Z}} s(t - kN).$$

Then $\widehat{s_N} : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ is naturally related to $\widehat{s} : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$:

$$\begin{aligned}
\widehat{s_N}(\nu) &= \sum_{t=1}^{N} \mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu/N}\, s_N(t) \\
&= \sum_{t=1}^{N} \mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu/N} \sum_{k\in\mathbb{Z}} s(t - kN) \\
&= \sum_{u\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}2\pi u\cdot\nu/N}\, s(u) \;=\; \widehat{s}(\nu/N).
\end{aligned}$$

Hence $\widehat{s_N}(\nu) = \widehat{s}(\nu/N)$ for all ν.

DFT related to Fourier series/coefficients (D1 vs. A1)

For "nice" periodic $s : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$, define $s_N : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ by

$$s_N(t) := s(t/N).$$

Then $\widehat{s_N} : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ is naturally related to $\widehat{s} : \mathbb{Z} \to \mathbb{C}$:

$$\begin{aligned}
\widehat{s_N}(\nu) &= \sum_{t=1}^{N} \mathrm{e}^{-\mathrm{i}2\pi t \cdot \nu/N}\ s(t/N) \\
&= \sum_{t=1}^{N} \mathrm{e}^{-\mathrm{i}2\pi t \cdot \nu/N} \sum_{\alpha \in \mathbb{Z}} \widehat{s}(\alpha)\ \mathrm{e}^{+\mathrm{i}2\pi (t/N) \cdot \alpha} \\
&= \sum_{\alpha \in \mathbb{Z}} \widehat{s}(\alpha) \sum_{t=1}^{N} \mathrm{e}^{\mathrm{i}2\pi t \cdot (\alpha - \nu)/N} \ = \ N \sum_{k \in \mathbb{Z}} \widehat{s}(\nu - kN).
\end{aligned}$$

Hence $\widehat{s_N}(\nu) = N \sum_{k \in \mathbb{Z}} \widehat{s}(\nu - kN)$ for all $\nu \in \mathbb{Z}$.

FFT (Fast Fourier Transform)...

FFT (Fast Fourier Transform) is a fast method for computing DFT. It is a divide-and-conquer algorithm, one of the most important tools in engineering and applied mathematics. Idea: Given signal $s : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, we want to find $F_N s = \widehat{s} : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, where $N = 2^k$. Split computation into two smaller size DFTs:

$$
\begin{aligned}
F_N s(\nu) &= \sum_{t=1}^{N} \mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu/N}\, s(t) \\
&= \sum_{t\in\{1,3,5,\cdots,N-1\}} \mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu/N}\, s(t) + \sum_{t\in\{2,4,6,\cdots,N\}} \mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu/N}\, s(t) \\
&= \sum_{t=1}^{N/2} \mathrm{e}^{-\mathrm{i}2\pi(2t-1)\cdot\nu/N} s(2t-1) + \sum_{t=1}^{N/2} \mathrm{e}^{-\mathrm{i}2\pi(2t)\cdot\nu/N} s(2t) \\
&= \mathrm{e}^{+\mathrm{i}2\pi\nu/N}\, F_{N/2} s_{\mathrm{Odd}}(\nu) + F_{N/2} s_{\mathrm{Even}}(\nu).
\end{aligned}
$$

Hence we just need to calculate $F_{N/2} s_{\mathrm{Odd}}$ and $F_{N/2} s_{\mathrm{Even}}$...

Why FFT requires only about $N\log(N)$ time units?

We say that the **complexity** of algorithm F_N is the "essential number" M_N of multiplications needed in computation. Obviously

$$F_N s(\nu) = \sum_{t=1}^{N} \mathrm{e}^{-\mathrm{i}2\pi t\cdot\nu/N}\; s(t)$$

yields $M_1 = 1$ and $M_N \le N^2$. However,

$$F_N s(\nu) = \mathrm{e}^{+\mathrm{i}2\pi\nu/N}\, F_{N/2} s_{\mathrm{Odd}}(\nu) + F_{N/2} s_{\mathrm{Even}}(\nu) \qquad (39)$$

implies recursively

$$\begin{aligned}
M_N &\overset{(39)}{\le} N + 2\,M_{N/2} \\
&\overset{(39)}{\le} N + 2(N/2 + 2\,M_{N/4}) \;=\; 2\,N + 4\,M_{N/4} \\
&\overset{(39)}{\le} 2\,N + 4(N/4 + 2\,M_{N/8}) \;=\; 3\,N + 8\,M_{N/8} \\
\cdots &\overset{(39)}{\le} \log_2(N)\,N + N\,M_{N/N} \;=\; N\log(N) + N \;\approx\; N\log(N).
\end{aligned}$$

Fast convolution via FFT

Direct calculation of discrete convolution $r * s : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ of signals $r, s : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ would require N^2 multiplications, as

$$r * s(t) = \sum_{u=1}^{N} r(t-u)\, s(u).$$

However,

$$\widehat{r * s}(\nu) = \widehat{r}(\nu)\, \widehat{s}(\nu),$$

where finding $\widehat{r}\,\widehat{s}$ takes only N multiplications. Computing each of

$$r \mapsto \widehat{r}, \quad s \mapsto \widehat{s}, \quad \widehat{\widehat{r}\widehat{s}} \mapsto r * s$$

takes only about $N \log(N)$ multiplications by FFT. Thus, computation $(r, s) \mapsto r * s$ has essential complexity $N \log(N)$, too!

Matlab computing FFT

Matlab command fft (Fast Fourier Transform) works as follows: vector $\mathtt{X} = \mathtt{fft}(\mathtt{x})$ for vector $\mathtt{x} = [\mathtt{x}(1)\ \mathtt{x}(2)\ \ldots\ \mathtt{x}(\mathtt{N})]$ is given by

$$\mathtt{X}(\mathtt{m}) = \sum_{\mathtt{k}=1}^{\mathtt{N}} \mathrm{e}^{-\mathrm{i}2\pi(\mathtt{k}-1)(\mathtt{m}-1)/\mathtt{N}}\ \mathtt{x}(\mathtt{k}), \tag{40}$$

instead of our more natural definition

$$\widehat{\mathtt{x}}(m) := \sum_{\mathtt{k}=1}^{\mathtt{N}} \mathrm{e}^{-\mathrm{i}2\pi\mathtt{k}\cdot\mathtt{m}/N}\ \mathtt{x}(\mathtt{k}). \tag{41}$$

That is, Matlab shifts both time and frequency by 1 always, and such a weird definition does not match well e.g. with convolution!
So, you have been warned!!!
Otherwise, Matlab is fine for computational Fourier analysis.

Time-frequency analysis

(**Remark!** A course on time-frequency analysis starts in April, 2017: more information on the last page!)

Next we try to understand behavior of signals simultaneously in both time and frequency. Applications of such time-frequency analysis include audio signal processing (phonetics, treating speech defects, speech synthesis, analyzing animal sounds, music), medical visualizations of EEG and ECG (ElectroEncephaloGraphy and ElectroCardioGraphy), sonar and radar imaging, seismology, quantum physics etc.
A time-frequency distribution for signal $s : \mathbb{R} \to \mathbb{C}$ is typically

$$Cs : \mathbb{R} \times \mathbb{R} \to \mathbb{C},$$

where $Cs(t, \nu)$ is "energy density of s at time-frequency (t, ν)".
There are many different time-frequency distributions to choose from, notably members of Leon Cohen's class, which includes e.g. all spectrograms and so-called Born–Jordan distribution.

Windowed Fourier Transform (STFT, Short-Time Fourier Transform)

For signals $s, w : \mathbb{R} \to \mathbb{C}$, w-**windowed Fourier transform** (**STFT**, Short-Time Fourier Transform) $F(s, w) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is

$$F(s, w)(t, \nu) := \widehat{s\,\overline{w_t}}(\nu), \tag{42}$$

where $w_t(u) = w(u - t)$. That is,

$$F(s, w)(t, \nu) = \int_{\mathbb{R}} s(u)\,\overline{w(u - t)}\,\mathrm{e}^{-\mathrm{i}2\pi u \cdot \nu}\,\mathrm{d}u.$$

Idea: Fourier transform $\widehat{s}(\nu)$ measures "content" of s at frequency $\nu \in \mathbb{R}$ over all times. $F(s, w)(t, \nu)$ measures "content" of s at time-frequency $(t, \nu) \in \mathbb{R} \times \mathbb{R}$ (when viewing s through window w).

Spectrogram (Sonogram)

Spectrogram related to the w-windowed Fourier transform is

$$|F(s,w)|^2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+. \tag{43}$$

Idea: $|F(s,w)(t,\nu)|^2 \geq 0$ is the "energy density" of signal $s : \mathbb{R} \to \mathbb{C}$ at time-frequency $(t,\nu) \in \mathbb{R} \times \mathbb{R}$ (when viewing s through window w).
For signal $s : \mathbb{R} \to \mathbb{C}$, choosing window w influences heavily the corresponding w-STFT and w-spectrogram!
In `Matlab`, try experimenting:

```
help spectrogram
```

Or, program your own spectrogram as in Exercises, implementing

$$|F(s,w)(t,\nu)|^2 = \left| \int_{\mathbb{R}} s(u)\, \overline{w(u-t)}\, \mathrm{e}^{-\mathrm{i}2\pi u \cdot \nu}\, \mathrm{d}u \right|^2.$$

Born–Jordan time-frequency distribution

For signals $r, s : \mathbb{R} \to \mathbb{C}$, the **Born–Jordan transform** $Q(r,s) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ is defined by

$$
\begin{aligned}
Q(r,s)(t,\nu) &:= \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi u\cdot\nu}\ \frac{1}{u} \int_{t-u/2}^{t+u/2} r(z+u/2)\ \overline{s(z-u/2)}\ \mathrm{d}z\ \mathrm{d}u \\
&= \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi u\cdot\nu}\ \frac{1}{u} \int_{t}^{t+u} r(z)\ \overline{s(z-u)}\ \mathrm{d}z\ \mathrm{d}u.
\end{aligned}
$$

The **Born–Jordan distribution** of $s : \mathbb{R} \to \mathbb{C}$ is

$$
Qs = Q(s) := Q(s,s) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}. \tag{44}
$$

Interpretation: $Qs(t,\nu) \in \mathbb{R}$ is the “energy density” of $s : \mathbb{R} \to \mathbb{C}$ at time-frequency $(t,\nu) \in \mathbb{R} \times \mathbb{R}$. Warning: $Qs(t,\nu)$ can be negative, but remember, that single points (t,ν) in time-frequency plane do not carry a physical meaning! Averaging Qs over larger areas yields positive values.

Properties of Born–Jordan distribution

Marginals: $\int_{\mathbb{R}} Qs(t,\nu)\,\mathrm{d}t = |\widehat{s}(\nu)|^2$, $\int_{\mathbb{R}} Qs(t,\nu)\,\mathrm{d}\nu = |s(t)|^2$.

Thus energy $\int_{\mathbb{R}}\int_{\mathbb{R}} Qs(t,\nu)\,\mathrm{d}t\,\mathrm{d}\nu = \|s\|^2$.

Natural Fourier symmetries: $Q\widehat{s}(\nu,t) = Qs(-t,\nu)$.

If $r(t) := s(t-t_0)$ and $q(t) := \mathrm{e}^{\mathrm{i}2\pi t\cdot\nu_0}\, s(t)$ then

$$\begin{aligned} Qr(t,\nu) &= Qs(t-t_0,\nu), \\ Qq(t,\nu) &= Qs(t,\nu-\nu_0). \end{aligned}$$

$Q\delta_{t_0}(t,\nu) = \delta_{t_0}(t)$.

$Qe_{\nu_0}(t,\nu) = \delta_{\nu_0}(\nu)$, where $e_{\nu_0}(t) := \mathrm{e}^{\mathrm{i}2\pi t\cdot\nu_0}$.

For $\alpha < \beta$: $Q(\lambda e_\alpha + \mu e_\beta)(t,\nu) =$

$$|\lambda|^2\delta_\alpha(\nu) + |\mu|^2\delta_\beta(\nu) + 2\,\mathrm{Re}\,(\lambda\,\overline{\mu}\,e_{\alpha-\beta}(t))\,\frac{\mathbf{1}_{[\alpha,\beta]}(\nu)}{\beta-\alpha}.$$

Born–Jordan filter design...

A **time-frequency symbol** is function $\sigma : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$. Now we design an integral operator A_σ such that we get a "best possible Born–Jordan approximation"

$$Q(A_\sigma s)(t,\nu) \approx \sigma(t,\nu)\, Qs(t,\nu)$$

for all signals $s : \mathbb{R} \to \mathbb{C}$ and for all $(t,\nu) \in \mathbb{R} \times \mathbb{R}$. Namely,

$$\langle r, A_\sigma s\rangle = \langle Q(r,s), \sigma\rangle \tag{45}$$

for all signals $r, s : \mathbb{R} \to \mathbb{C}$: here $\langle r, A_\sigma s\rangle = \langle Q(r,s), \sigma\rangle =$

$$\begin{aligned}
&= \int_{\mathbb{R}} \int_{\mathbb{R}} Q(r,s)(z,\nu)\, \overline{\sigma(z,\nu)}\, \mathrm{d}z\, \mathrm{d}\nu \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}2\pi w\cdot\nu} \frac{1}{w} \int_{z-\frac{w}{2}}^{z+\frac{w}{2}} r(\tilde{t}+\frac{w}{2})\overline{s(\tilde{t}-\frac{w}{2})}\, \mathrm{d}\tilde{t}\, \mathrm{d}w\, \overline{\sigma(z,\nu)}\, \mathrm{d}z\, \mathrm{d}\nu \\
&= \int_{\mathbb{R}} r(t) \left[\int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi(t-u)\cdot\nu} s(u) \frac{1}{u-t} \int_t^u \sigma(z,\nu)\, \mathrm{d}z\, \mathrm{d}u\, \mathrm{d}\nu \right]^* \mathrm{d}t.
\end{aligned}$$

... Hence

$$A_\sigma s(t) = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi(t-u)\cdot\nu} s(u)\ a(t,u,\nu)\ \mathrm{d}u\ \mathrm{d}\nu, \tag{46}$$

where $a(t,t,\nu) = \sigma(t,\nu)$, and for $t \neq u$ we have amplitude

$$a(t,u,\nu) = \frac{1}{u-t} \int_t^u \sigma(z,\nu)\ \mathrm{d}z. \tag{47}$$

We obtained

$$A_\sigma s(t) = \int_{\mathbb{R}} K_{A_\sigma}(t,u)\ s(u)\ \mathrm{d}u,$$

where kernel $K_{A_\sigma} : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ of integral operator A_σ is given by $K_{A_\sigma}(t,t) = \int_{\mathbb{R}} \sigma(t,\nu)\ \mathrm{d}\nu$, and for $t \neq u$ by

$$K_{A_\sigma}(t,u) = \frac{1}{u-t} \int_t^u \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi(t-u)\cdot\nu}\ \sigma(z,\nu)\ \mathrm{d}\nu\ \mathrm{d}z. \tag{48}$$

Filtering examples

On previous page, suppose time-invariance $\sigma(t,\nu) = \widehat{\psi}(\nu)$ for all $(t,\nu) \in \mathbb{R} \times \mathbb{R}$. Naturally, then $A_\sigma s = \psi * s$, because

$$\begin{aligned}
A_\sigma s(t) &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi(t-u)\cdot\nu}\ s(u)\ \frac{1}{u-t} \int_t^u \widehat{\psi}(\nu)\ \mathrm{d}z\ \mathrm{d}u\ \mathrm{d}\nu \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi(t-u)\cdot\nu}\ s(u)\ \widehat{\psi}(\nu)\ \mathrm{d}u\ \mathrm{d}\nu \\
&= \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi t\cdot\nu}\ \widehat{s}(\nu)\ \widehat{\psi}(\nu)\ \mathrm{d}\nu \ = \ \psi * s(t).
\end{aligned}$$

On previous page, suppose frequency-invariance $\sigma(t,\nu) = \varphi(t)$ for all $(t,\nu) \in \mathbb{R} \times \mathbb{R}$. Then $a(t,u,\nu) = b(t,u)$ so that $A_\sigma s = \varphi\, s$:

$$\begin{aligned}
A_\sigma s(t) &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}2\pi(t-u)\cdot\nu}\ s(u)\ b(t,u)\ \mathrm{d}u\ \mathrm{d}\nu \\
&= s(t)\ b(t,t) \ = \ \varphi(t)\, s(t).
\end{aligned}$$

Time-limited signal which is band-limited, too?

Let $\|s\|^2 < \infty$, where $s : \mathbb{R} \to \mathbb{C}$ is limited in time-frequency:

$$s(t) = 0 = \widehat{s}(\nu)$$

whenever $|t| > M$ and $|\nu| > M$ for some constant $M < \infty$.
Then define analytic function $h : \mathbb{C} \to \mathbb{C}$ by

$$h(z) := \int_{-M}^{M} \mathrm{e}^{-\mathrm{i}2\pi t\cdot z}\ s(t)\ \mathrm{d}t.$$

Due to analyticity, for any $a \in \mathbb{C}$ we have power series

$$h(z) = \sum_{k=0}^{\infty} \frac{1}{k!}\ h^{(k)}(a)\ (z-a)^k.$$

If $M < a \in \mathbb{R}$ then $h(a) = \widehat{s}(a) = 0$, yielding $h(z) \equiv 0$ for all $z \in \mathbb{C}$.
But here $\widehat{s}(\nu) = h(\nu) \equiv 0$ for all $\nu \in \mathbb{R}$, so $s(t) \equiv 0$ for all $t \in \mathbb{R}$.
[Remark: Schwartz test functions $s \in \mathscr{S}(\mathbb{R}) \subset C^\infty(\mathbb{R})$
(e.g. Gaussian signals) are "almost time- and frequency-limited",
because $s(t), \widehat{s}(t) \to 0$ rapidly as $|t| \to \infty$.]

Heat flow: historical origin of Fourier analysis

Let $u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ satisfy so-called **heat equation**

$$\frac{\partial}{\partial t} u(x,t) = \alpha \left(\frac{\partial}{\partial x} \right)^2 u(x,t), \qquad (49)$$

with initial condition $u(x,0) = f(x)$, where $\alpha > 0$ is the thermal diffusivity constant. Here $u_t(x) = u(x,t)$ is "temperature at point x at time t". Taking Fourier transform in the x-variable, we get

$$\frac{\partial}{\partial t} \widehat{u_t}(\xi) = -(2\pi\xi)^2 \alpha \; \widehat{u_t}(\xi) \quad \text{and} \quad \widehat{u_0}(\xi) = \widehat{f}(\xi),$$

so that

$$\begin{aligned} \widehat{u_t}(\xi) &= \mathrm{e}^{-(2\pi\xi)^2 \alpha t} \; \widehat{f}(\xi), \\ u(x,t) &= \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i} 2\pi x \cdot \xi} \; \mathrm{e}^{-(2\pi\xi)^2 \alpha t} \; \widehat{f}(\xi) \, \mathrm{d}\xi. \end{aligned}$$

Fourier found this reasoning for periodic x case in 1807, but already Daniel Bernoulli and Leonhard Euler considered vibrating strings as trigonometric series in 1753; and Gauss invented FFT in 1805.

Review: how are different Fourier transforms related?

Time space G (continuous $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$; discrete $\mathbb{Z}$ and $\mathbb{Z}/N\mathbb{Z}$).
Frequency space $\widehat{G}$ is dual to the time space G.
Signal $s : G \to \mathbb{C}$ has Fourier transform $\widehat{s} : \widehat{G} \to \mathbb{C}$,

$$\begin{aligned} \widehat{s}(\nu) &= \int_G \mathrm{e}^{-\mathrm{i}\langle t,\nu\rangle} \, s(t) \, \mathrm{d}t, \\ s(t) &= \int_{\widehat{G}} \mathrm{e}^{+\mathrm{i}\langle t,\nu\rangle} \, \widehat{s}(\nu) \, \mathrm{d}\nu, \end{aligned}$$

"energy conservation" $\|\widehat{s}\|^2 = \|s\|^2$ (except for DFT), where

$$\|s\|^2 = \int_G |s(t)|^2 \, \mathrm{d}t$$

(for DFT, energy conservation needed a constant...).
Convolution $r * s : G \to \mathbb{C}$ of signals $r, s : G \to \mathbb{C}$,

$$r * s(t) = \int_G r(t-u) \, s(u) \, \mathrm{d}u,$$

which can be in finite case computed efficiently by FFT.

Review problems and questions

In your field of science/engineering, find examples of signals

$$s : \mathbb{R} \to \mathbb{C}, \qquad s : \mathbb{R}/\mathbb{Z} \to \mathbb{C},$$
$$s : \mathbb{Z} \to \mathbb{C}, \qquad s : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}.$$

In each of these cases:

- How is Fourier transform defined? Which kind of signal is it?
- How is energy defined? Interpretation of energy?
- How does the inverse Fourier transform look like?
- How is convolution defined? Applications to signal processing?

How are these different Fourier transforms related to each other?
Why is FFT fast?
What do time-frequency distributions tell us?

Advertisement: follow-up course for Fourier analysis!

MS-E1993 **Time-Frequency Analysis** (5 credits)
in period V of Spring 2017: starts on April 11 (or April 12).

Learning outcomes: We shall learn modern methods for analyzing and processing signals. Time-frequency analysis is a subfield of Fourier analysis, studying simultaneously WHEN and HOW OFTEN something happens in a signal. With sharp time-frequency localizations we can apply sharp time-frequency operations to signals. The course is meant for students in all fields of engineering, science and mathematics.

Content: Symmetries of time and frequency, Heisenberg group. Uncertainty principle in Fourier analysis. Short-time Fourier transform and spectrograms. Quadratic time-frequency transforms. Time-frequency localization and time-varying filters. Discretization and computation.