5

The Variational Approach to Optimal Control Problems

In this chapter we shall apply variational methods to optimal control problems. We shall first derive necessary conditions for optimal control assuming that the admissible controls are not bounded. These necessary conditions are then employed to find the optimal control law for the important linear regulator problem. Next, Pontryagin's minimum principle is introduced heuristically as a generalization of the fundamental theorem of the calculus of variations, and problems with bounded control and state variables are discussed. The three concluding sections of the chapter are devoted to time-optimal problems, minimum control-effort systems, and problems involving singular intervals.

5.1 NECESSARY CONDITIONS FOR OPTIMAL CONTROL

Let us now employ the techniques introduced in Chapter 4 to determine necessary conditions for optimal control. As stated in Chapter 1, the problem is to find an admissible control $\mathbf{u}^*$ that causes the system

$$\dot{\mathbf{x}}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t) \tag{5.1-1}$$

to follow an admissible trajectory $\mathbf{x}^*$ that minimizes the performance measure

$$J(\mathbf{u}) = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t)\, dt.\dagger \tag{5.1-2}$$

We shall initially assume that the admissible state and control regions are not bounded, and that the initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$ and the initial time t_0 are specified. As usual, $\mathbf{x}$ is the $n \times 1$ state vector and $\mathbf{u}$ is the $m \times 1$ vector of control inputs.

In the terminology of Chapter 4, we have a problem involving $n + m$ functions which must satisfy the n differential equation constraints (5.1-1). The m control inputs are the independent functions.

The only difference between Eq. (5.1-2) and the functionals considered in Chapter 4 is the term involving the final states and final time. However, assuming that h is a differentiable function, we can write

$$h(\mathbf{x}(t_f), t_f) = \int_{t_0}^{t_f} \frac{d}{dt}[h(\mathbf{x}(t), t)]\, dt + h(\mathbf{x}(t_0), t_0), \tag{5.1-3}$$

so that the performance measure can be expressed as

$$J(\mathbf{u}) = \int_{t_0}^{t_f} \left\{ g(\mathbf{x}(t), \mathbf{u}(t), t) + \frac{d}{dt}[h(\mathbf{x}(t), t)] \right\} dt + h(\mathbf{x}(t_0), t_0). \tag{5.1-4}$$

Since $\mathbf{x}(t_0)$ and t_0 are fixed, the minimization does not affect the $h(\mathbf{x}(t_0), t_0)$ term, so we need consider only the functional

$$J(\mathbf{u}) = \int_{t_0}^{t_f} \left\{ g(\mathbf{x}(t), \mathbf{u}(t), t) + \frac{d}{dt}[h(\mathbf{x}(t), t)] \right\} dt. \tag{5.1-5}$$

Using the chain rule of differentiation, we find that this becomes

$$J(\mathbf{u}) = \int_{t_0}^{t_f} \left\{ g(\mathbf{x}(t), \mathbf{u}(t), t) + \left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}(t), t)\right]^T \dot{\mathbf{x}}(t) + \frac{\partial h}{\partial t}(\mathbf{x}(t), t) \right\} dt. \tag{5.1-6}$$

To include the differential equation constraints, we form the augmented functional

$$\begin{aligned} J_a(\mathbf{u}) = \int_{t_0}^{t_f} \bigg\{ & g(\mathbf{x}(t), \mathbf{u}(t), t) + \left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}(t), t)\right]^T \dot{\mathbf{x}}(t) + \frac{\partial h}{\partial t}(\mathbf{x}(t), t) \\ & + \mathbf{p}^T(t)[\mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t) - \dot{\mathbf{x}}(t)] \bigg\} dt \end{aligned} \tag{5.1-7}$$

by introducing the Lagrange multipliers $p_1(t), \ldots, p_n(t)$. Let us define

† In general, the functional J depends on $\mathbf{x}(t_0)$, t_0, $\mathbf{x}$, $\mathbf{u}$, the target set S, and t_f. However, here it is assumed that $\mathbf{x}(t_0)$ and t_0 are specified; hence, $\mathbf{x}$ is determined by $\mathbf{u}$ and we write $J(\mathbf{u})$—the dependence of J on S and t_f will not be explicitly indicated.

$$g_a(\mathbf{x}(t), \dot{\mathbf{x}}(t), \mathbf{u}(t), \mathbf{p}(t), t) \triangleq g(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}^T(t)[\mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t) - \dot{\mathbf{x}}(t)] + \left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}(t), t)\right]^T \dot{\mathbf{x}}(t) + \frac{\partial h}{\partial t}(\mathbf{x}(t), t)$$

so that

$$J_a(\mathbf{u}) = \int_{t_0}^{t_f} \{g_a(\mathbf{x}(t), \dot{\mathbf{x}}(t), \mathbf{u}(t), \mathbf{p}(t), t)\}\, dt. \tag{5.1-8}$$

We shall assume that the end points at $t = t_f$ can be specified or free. To determine the variation of J_a, we introduce the variations $\delta\mathbf{x}$, $\delta\dot{\mathbf{x}}$, $\delta\mathbf{u}$, $\delta\mathbf{p}$, and δt_f. From *Problem 4a* in the preceding chapter this gives [see Eq. (4.3-16)], on an extremal,

$$\begin{aligned}
\delta J_a(\mathbf{u}^*) = 0 = &\left[\frac{\partial g_a}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f)\right]^T \delta\mathbf{x}_f \\
&+ \Big[g_a(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) \\
&- \left[\frac{\partial g_a}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t_f), \dot{\mathbf{x}}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f)\right]^T \dot{\mathbf{x}}^*(t_f)\Big] \delta t_f \\
&+ \int_{t_0}^{t_f} \Big\{\Big[\left[\frac{\partial g_a}{\partial \mathbf{x}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)\right]^T \\
&- \frac{d}{dt}\left[\frac{\partial g_a}{\partial \dot{\mathbf{x}}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)\right]^T\Big] \delta\mathbf{x}(t) \\
&+ \left[\frac{\partial g_a}{\partial \mathbf{u}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)\right]^T \delta\mathbf{u}(t) \\
&+ \left[\frac{\partial g_a}{\partial \mathbf{p}}(\mathbf{x}^*(t), \dot{\mathbf{x}}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t)\right]^T \delta\mathbf{p}(t)\Big\}\, dt.
\end{aligned} \tag{5.1-9}$$

Notice that the above result is obtained because $\dot{\mathbf{u}}(t)$ and $\dot{\mathbf{p}}(t)$ do not appear in g_a.

Next, let us consider only those terms inside the integral which involve the function h; these terms contain

$$\frac{\partial}{\partial \mathbf{x}}\left[\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t), t)\right]^T \dot{\mathbf{x}}^*(t) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t), t)\right] - \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{\mathbf{x}}}\left[\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t), t)\right]^T \dot{\mathbf{x}}^*(t)\right]\right\}. \tag{5.1-10}$$

Writing out the indicated partial derivatives gives

$$\left[\frac{\partial^2 h}{\partial \mathbf{x}^2}(\mathbf{x}^*(t), t)\right]\dot{\mathbf{x}}^*(t) + \left[\frac{\partial^2 h}{\partial t \partial \mathbf{x}}(\mathbf{x}^*(t), t)\right] - \frac{d}{dt}\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t), t)\right], \tag{5.1-11}$$

or, if we apply the chain rule to the last term,

$$\left[\frac{\partial^2 h}{\partial \mathbf{x}^2}(\mathbf{x}^*(t), t)\right]\dot{\mathbf{x}}^*(t) + \left[\frac{\partial^2 h}{\partial t \partial \mathbf{x}}(\mathbf{x}^*(t), t)\right] - \left[\frac{\partial^2 h}{\partial \mathbf{x}^2}(\mathbf{x}^*(t), t)\right]\dot{\mathbf{x}}^*(t)$$
$$- \left[\frac{\partial^2 h}{\partial \mathbf{x} \partial t}(\mathbf{x}^*(t), t)\right]. \tag{5.1-12}$$

If it is assumed that the second partial derivatives are continuous, the order of differentiation can be interchanged, and these terms add to zero. In the integral term we have, then,

$$\int_{t_0}^{t_f} \left\{\left[\left[\frac{\partial g}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)\right]^T + \mathbf{p}^{*T}(t)\left[\frac{\partial \mathbf{a}}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)\right]\right.\right.$$
$$\left. - \frac{d}{dt}\left[-\mathbf{p}^{*T}(t)\right]\right]\delta\mathbf{x}(t) + \left[\left[\frac{\partial g}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)\right]^T\right.$$
$$\left.\left. + \mathbf{p}^{*T}(t)\left[\frac{\partial \mathbf{a}}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)\right]\right]\delta\mathbf{u}(t) + \left[\left[\mathbf{a}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) - \dot{\mathbf{x}}^*(t)\right]^T\right]\delta\mathbf{p}(t)\right\} dt.$$
$$\tag{5.1-13}$$

This integral must vanish on an extremal regardless of the boundary conditions. We first observe that the constraints

$$\dot{\mathbf{x}}^*(t) = \mathbf{a}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) \tag{5.1-14a}$$

must be satisfied by an extremal so that the coefficient of $\delta\mathbf{p}(t)$ is zero. The Lagrange multipliers are arbitrary, so let us select them to make the coefficient of $\delta\mathbf{x}(t)$ equal to zero, that is,

$$\dot{\mathbf{p}}^*(t) = -\left[\frac{\partial \mathbf{a}}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)\right]^T \mathbf{p}^*(t) - \frac{\partial g}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t). \tag{5.1-14b}$$

We shall henceforth call (5.1-14b) the *costate equations* and $\mathbf{p}(t)$ the *costate.*

The remaining variation $\delta\mathbf{u}(t)$ is independent, so its coefficient must be zero; thus,

$$\mathbf{0} = \frac{\partial g}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t) + \left[\frac{\partial \mathbf{a}}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), t)\right]^T \mathbf{p}^*(t). \tag{5.1-14c}$$

Equations (5.1-14) are important equations; we shall be using them throughout the remainder of this chapter. We shall find that even when the admissible controls are bounded, only Eq. (5.1-14c) is modified.

There are still the terms outside the integral to deal with; since the variation must be zero, we have

$$\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f)\right]^T \delta\mathbf{x}_f + \left[g(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f)\right.$$
$$\left. + \mathbf{p}^{*T}(t_f)[\mathbf{a}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), t_f)]\right]\delta t_f = 0. \tag{5.1-15}$$

In writing (5.1-15), we have used the fact that $\dot{\mathbf{x}}^*(t_f) = \mathbf{a}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), t_f)$. Equation (5.1-15) admits a variety of situations, which we shall discuss shortly.

Equations (5.1-14) are the necessary conditions we set out to determine. Notice that these necessary conditions consist of a set of $2n$, *first-order* differential equations—the state and costate equations (5.1-14a) and (5.1-14b)—and a set of m algebraic relations—(5.1-14c)—which must be satisfied throughout the interval $[t_0, t_f]$. The solution of the state and costate equations will contain $2n$ constants of integration. To evaluate these constants we use the n equations $\mathbf{x}^*(t_0) = \mathbf{x}_0$ and an additional set of n or $(n+1)$ relationships —depending on whether or not t_f is specified—from Eq. (5.1-15). Notice that, as expected, we are again confronted by a two-point boundary-value problem.

In the following we shall find it convenient to use the function $\mathscr{H}$, called the *Hamiltonian*, defined as

$$\mathscr{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}(t), t) \triangleq g(\mathbf{x}(t), \mathbf{u}(t), t) + \mathbf{p}^T(t)[\mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t)]. \qquad (5.1\text{-}16)$$

Using this notation, we can write the necessary conditions (5.1-14) through (5.1-15) as follows:

$$\boxed{\left.\begin{aligned} \dot{\mathbf{x}}^*(t) &= \frac{\partial \mathscr{H}}{\partial \mathbf{p}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \\ \dot{\mathbf{p}}^*(t) &= -\frac{\partial \mathscr{H}}{\partial \mathbf{x}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \\ \mathbf{0} &= \frac{\partial \mathscr{H}}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \mathbf{p}^*(t), t) \end{aligned}\right\} \begin{array}{l}\text{for all} \\ t \in [t_0, t_f]\end{array}} \qquad \begin{array}{r}(5.1\text{-}17\text{a}) \\ (5.1\text{-}17\text{b}) \\ (5.1\text{-}17\text{c})\end{array}$$

and

$$\boxed{\begin{aligned} &\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f)\right]^T \delta\mathbf{x}_f + \Big[\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) \\ &\qquad + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f)\Big]\, \delta t_f = 0. \end{aligned}} \qquad (5.1\text{-}18)$$

Let us now consider the boundary conditions that may occur.

Boundary Conditions

In a particular problem either g or h may be missing; in this case, we simply strike out the terms involving the missing function. To determine the boundary conditions is a matter of making the appropriate substitutions in Eq. (5.1-18). In all cases it will be assumed that we have the n equations $\mathbf{x}^*(t_0) = \mathbf{x}_0$.

Problems with Fixed Final Time. If the final time t_f is specified, $\mathbf{x}(t_f)$ may be specified, free, or required to lie on some surface in the state space.

CASE I. *Final state specified.* Since $\mathbf{x}(t_f)$ and t_f are specified, we substitute $\delta\mathbf{x}_f = \mathbf{0}$ and $\delta t_f = 0$ in (5.1-18). The required n equations are

$$\mathbf{x}^*(t_f) = \mathbf{x}_f. \tag{5.1-19}$$

CASE II. *Final state free.* We substitute $\delta t_f = 0$ in Eq. (5.1-18); since $\delta\mathbf{x}_f$ is arbitrary, the n equations

$$\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) - \mathbf{p}^*(t_f) = \mathbf{0}\dagger \tag{5.1-20}$$

must be satisfied.

CASE III. *Final state lying on the surface defined by* $\mathbf{m}(\mathbf{x}(t)) = \mathbf{0}$. Since this is a new situation, let us consider an introductory example. Suppose that the final state of a second-order system is required to lie on the circle

$$m(\mathbf{x}(t)) = [x_1(t) - 3]^2 + [x_2(t) - 4]^2 - 4 = 0 \tag{5.1-21}$$

shown in Fig. 5-1. Notice that admissible changes in $\mathbf{x}(t_f)$ are (to first-order) tangent to the circle at the point $(\mathbf{x}^*(t_f), t_f)$. The tangent line is normal to the gradient vector

$$\frac{\partial m}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) = \begin{bmatrix} 2[x_1^*(t_f) - 3] \\ 2[x_2^*(t_f) - 4] \end{bmatrix} \tag{5.1-22}$$

at the point $(\mathbf{x}^*(t_f), t_f)$. Thus, $\delta\mathbf{x}(t_f)$ must be normal to the gradient (5.1-22), so that

$$\left[\frac{\partial m}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right]^T \delta\mathbf{x}(t_f) = 2[x_1^*(t_f) - 3]\,\delta x_1(t_f) + 2[x_2^*(t_f) - 4]\,\delta x_2(t_f) = 0. \tag{5.1-23}$$

† Since the final time is fixed, h will not depend on t_f.

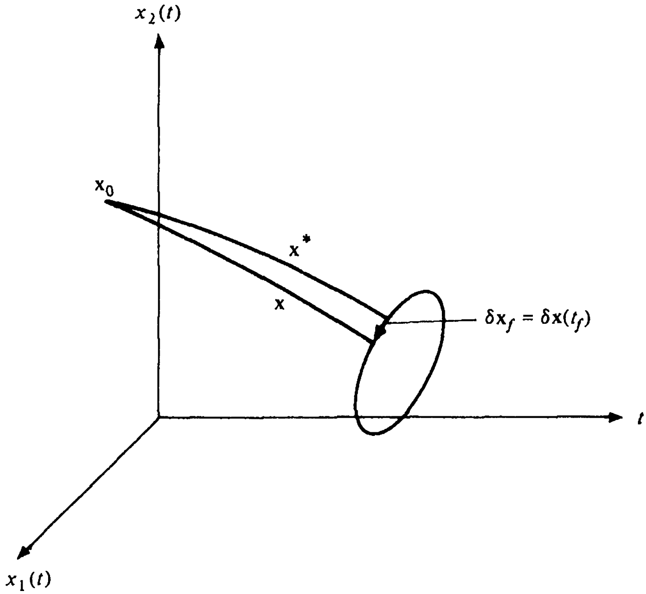

Figure 5-1 An extremal and a comparison curve that terminate on the curve $[x_1(t) - 3]^2 + [x_2(t) - 4]^2 - 4 = 0$ at the specified final time, t_f

Solving for $\delta x_2(t_f)$ gives

$$\delta x_2(t_f) = \frac{-[x_1^*(t_f) - 3]}{[x_2^*(t_f) - 4]}\,\delta x_1(t_f), \tag{5.1-24}$$

which, when substituted in Eq. (5.1-18), gives

$$\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) - \mathbf{p}^*(t_f)\right]^T \begin{bmatrix} 1 \\ \dfrac{-[x_1^*(t_f) - 3]}{[x_2^*(t_f) - 4]} \end{bmatrix} = 0 \tag{5.1-25}$$

since $\delta t_f = 0$ and $\delta x_1(t_f)$ is arbitrary. The second required equation at the final time is

$$m(\mathbf{x}^*(t_f)) = [x_1^*(t_f) - 3]^2 + [x_2^*(t_f) - 4]^2 - 4 = 0. \tag{5.1-26}$$

In the general situation there are n state variables and $1 \leq k \leq n - 1$ relationships that the states must satisfy at $t = t_f$. In this case we write

$$\mathbf{m}(\mathbf{x}(t)) = \begin{bmatrix} m_1(\mathbf{x}(t)) \\ \cdot \\ \cdot \\ \cdot \\ m_k(\mathbf{x}(t)) \end{bmatrix} = \mathbf{0}, \tag{5.1-27}$$

and each component of $\mathbf{m}$ represents a hypersurface in the n-dimensional state space. Thus, the final state lies on the intersection of these k hypersurfaces, and $\delta\mathbf{x}(t_f)$ is tangent to each of the hypersurfaces at the point $(\mathbf{x}^*(t_f), t_f)$. This means that $\delta\mathbf{x}(t_f)$ is normal to each of the gradient vectors

$$\frac{\partial m_1}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)), \ldots, \frac{\partial m_k}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)), \tag{5.1-28}$$

which are assumed to be linearly independent. From Eq. (5.1-18) we have, since $\delta t_f = 0$,

$$\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) - \mathbf{p}^*(t_f)\right]^T \delta\mathbf{x}(t_f) \triangleq \mathbf{v}^T \delta\mathbf{x}(t_f) = 0. \tag{5.1-29}$$

It can be shown that this equation is satisfied if and only if the vector $\mathbf{v}$ is a linear combination of the gradient vectors in Eq. (5.1-28), that is,

$$\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) - \mathbf{p}^*(t_f) = d_1\left[\frac{\partial m_1}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right] + \cdots + d_k\left[\frac{\partial m_k}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right]. \tag{5.1-30}$$

To determine the $2n$ constants of integration in the solution of the state-costate equations, and $d_1, \ldots, d_k$, we have the n equations $\mathbf{x}^*(t_0) = \mathbf{x}_0$, the n equations (5.1-30), and the k equations

$$\mathbf{m}(\mathbf{x}^*(t_f)) = \mathbf{0}. \tag{5.1-31}$$

Let us show that Eqs. (5.1-30) and (5.1-31) lead to the results obtained in our introductory example. The constraining relation is

$$m(\mathbf{x}(t)) = [x_1(t) - 3]^2 + [x_2(t) - 4]^2 - 4 = 0. \tag{5.1-21}$$

From Eq. (5.1-30) we obtain the two equations

$$\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) - \mathbf{p}^*(t_f) = d\begin{bmatrix} 2[x_1^*(t_f) - 3] \\ 2[x_2^*(t_f) - 4] \end{bmatrix}, \tag{5.1-32}$$

and (5.1-31) gives

$$m(\mathbf{x}^*(t_f)) = [x_1^*(t_f) - 3]^2 + [x_2^*(t_f) - 4]^2 - 4 = 0. \tag{5.1-33}$$

By solving the second of Eqs. (5.1-32) for d and substituting this into the first equation of (5.1-32), Eq. (5.1-25) is obtained.

Problems with Free Final Time. If the final time is free, there are several situations that may occur.

CASE I. *Final state fixed.* The appropriate substitution in Eq. (5.1-18) is $\delta\mathbf{x}_f = \mathbf{0}$. δt_f is arbitrary, so the $(2n + 1)$st relationship is

$$\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = 0. \tag{5.1-34}$$

CASE II. *Final state free.* $\delta\mathbf{x}_f$ and δt_f are arbitrary and independent; therefore, their coefficients must be zero; that is,

$$\mathbf{p}^*(t_f) = \frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) \qquad (n \text{ equations}) \tag{5.1-35}$$

$$\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = 0 \qquad (1 \text{ equation}). \tag{5.1-36}$$

Notice that if $h = 0$

$$\mathbf{p}^*(t_f) = \mathbf{0} \tag{5.1-37}$$

$$\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) = 0. \tag{5.1-38}$$

CASE III. $\mathbf{x}(t_f)$ *lies on the moving point* $\boldsymbol{\theta}(t)$. Here $\delta\mathbf{x}_f$ and δt_f are related by

$$\delta\mathbf{x}_f \doteq \left[\frac{d\boldsymbol{\theta}}{dt}(t_f)\right] \delta t_f;$$

making this substitution in Eq. (5.1-18) yields the equation

$$\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) + \left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f)\right]^T \times \left[\frac{d\boldsymbol{\theta}}{dt}(t_f)\right] = 0. \tag{5.1-39}$$

This gives one equation; the remaining n required relationships are

$$\mathbf{x}^*(t_f) = \boldsymbol{\theta}(t_f).$$

CASE IV. *Final state lying on the surface defined by* $\mathbf{m}(\mathbf{x}(t)) = \mathbf{0}$. As an example of this type of end point constraint, suppose that the final state is required to lie on the curve

$$m(\mathbf{x}(t)) = [x_1(t) - 3]^2 + [x_2(t) - 4]^2 - 4 = 0. \tag{5.1-40}$$

Since the final time is free, the admissible end points lie on the cylindrical surface shown in Fig. 5-2. Notice that

1. To first-order, the change in $\mathbf{x}(t_f)$ must be in the plane tangent to the cylindrical surface at the point $(\mathbf{x}^*(t_f), t_f)$.
2. The change in $\mathbf{x}(t_f)$ is independent of δt_f.

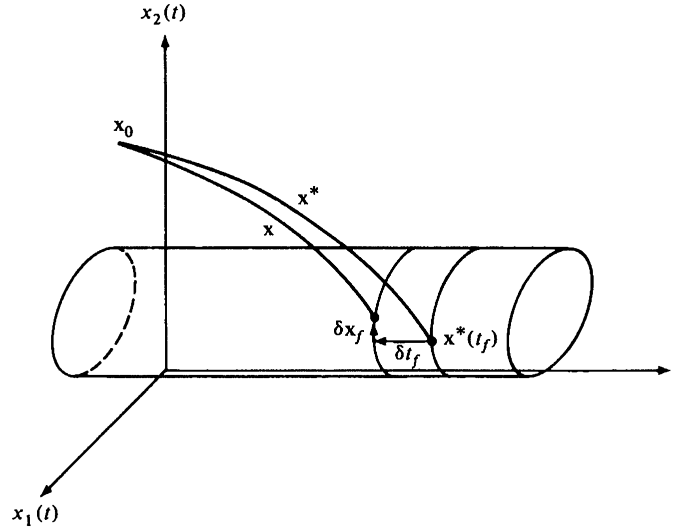

Figure 5-2 An extremal and a comparison curve that terminate on the surface $[x_1(t) - 3]^2 + [x_2(t) - 4]^2 - 4 = 0$

Since $\delta\mathbf{x}_f$ is independent of δt_f, the coefficient of δt_f must be zero, and

$$\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = 0. \tag{5.1-41}$$

The plane that is tangent to the cylinder at the point $(\mathbf{x}^*(t_f), t_f)$ is described by its normal vector or gradient; that is, every vector in the plane is normal to the vector

$$\frac{\partial m}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) = \begin{bmatrix} 2[x_1^*(t_f) - 3] \\ 2[x_2^*(t_f) - 4] \end{bmatrix}. \tag{5.1-42}$$

This means that

$$\left[\frac{\partial m}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right]^T \delta\mathbf{x}_f = 2[x_1^*(t_f) - 3]\,\delta x_{1_f} + 2[x_2^*(t_f) - 4]\,\delta x_{2_f} = 0. \quad (5.1\text{-}43)$$

Solving for δx_{2_f} gives

$$\delta x_{2_f} = \frac{-[x_1^*(t_f) - 3]}{[x_2^*(t_f) - 4]}\,\delta x_{1_f}. \quad (5.1\text{-}44)$$

Substituting this for δx_{2_f} in Eq. (5.1-18) gives

$$\left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f)\right]^T \begin{bmatrix} 1 \\ \dfrac{-[x_1^*(t_f) - 3]}{[x_2^*(t_f) - 4]} \end{bmatrix} \delta x_{1_f} = 0. \quad (5.1\text{-}45)$$

Since δx_{1_f} is arbitrary, its coefficient must be zero. Equations (5.1-41) and (5.1-45) give two relationships; the third is the constraint

$$m(\mathbf{x}^*(t_f)) = [x_1^*(t_f) - 3]^2 + [x_2^*(t_f) - 4]^2 - 4 = 0. \quad (5.1\text{-}46)$$

In the general situation we have n state variables, and there may be $1 \leq k \leq n - 1$ relationships that the states are required to satisfy at the terminal time. In this case we write

$$\mathbf{m}(\mathbf{x}(t)) = \begin{bmatrix} m_1(\mathbf{x}(t)) \\ \cdot \\ \cdot \\ \cdot \\ m_k(\mathbf{x}(t)) \end{bmatrix} = \mathbf{0}, \quad (5.1\text{-}47)$$

and each component of $\mathbf{m}$ describes a hypersurface in the n-dimensional state space. This means that the final state lies on the intersection of the hypersurfaces defined by $\mathbf{m}$, and that $\delta\mathbf{x}_f$ is (to first order) tangent to each of the hypersurfaces at the point $(\mathbf{x}^*(t_f), t_f)$. Thus, $\delta\mathbf{x}_f$ is normal to each of the gradient vectors

$$\frac{\partial m_1}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)), \ldots, \frac{\partial m_k}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)), \quad (5.1\text{-}48)$$

which we assume to be linearly independent. It is left as an exercise for the reader to show that the reasoning used in Case III with *fixed* final time also applies in the present situation and leads to the $(2n + k + 1)$ equations

$$\mathbf{x}^*(t_0) = \mathbf{x}_0$$
$$\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f) = d_1\left[\frac{\partial m_1}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right] + \cdots + d_k\left[\frac{\partial m_k}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right]$$
$$\mathbf{m}(\mathbf{x}^*(t_f)) = \mathbf{0}$$
$$\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = 0 \tag{5.1-49}$$

involving the $2n$ constants of integration, the variables $d_1, \ldots, d_k$, and t_f. It is also easily shown that Eqs. (5.1-49) give Eqs. (5.1-41), (5.1-45), and (5.1-46) in the preceding example.

CASE V. *Final state lying on the moving surface defined by* $\mathbf{m}(\mathbf{x}(t), t) = \mathbf{0}$. Suppose that the final state must lie on the surface

$$m(\mathbf{x}(t), t) = [x_1(t) - 3]^2 + [x_2(t) - 4 - t]^2 - 4 = 0 \tag{5.1-50}$$

shown in Fig. 5-3. Notice that δt_f does influence the admissible values of $\delta \mathbf{x}_f$; that is, to remain on the surface $m(\mathbf{x}(t), t) = 0$ the value of $\delta \mathbf{x}_f$ depends on δt_f. The vector with components $\delta x_{1_f}, \delta x_{2_f}, \delta t_f$ must be contained in a plane tangent to the surface at the point $(\mathbf{x}^*(t_f), t_f)$. This means that the normal to this tangent plane is the vector

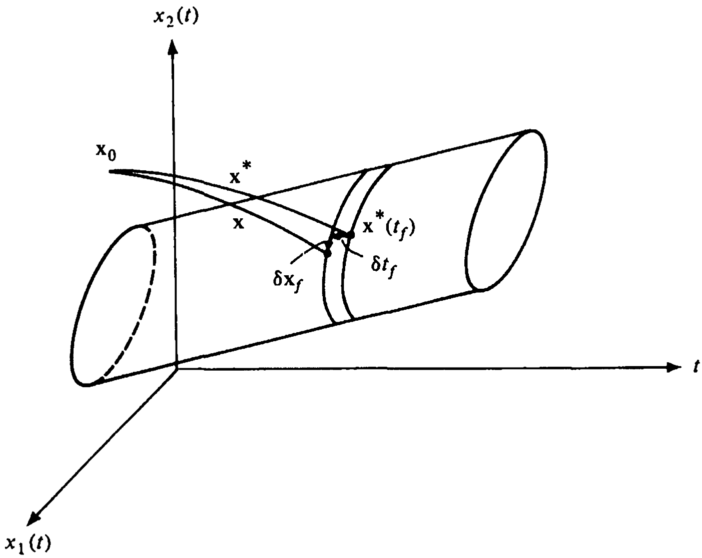

Figure 5-3 An extremal and a comparison curve that terminate on the surface $[x_1(t) - 3]^2 + [x_2(t) - 4 - t]^2 - 4 = 0$

$$\begin{bmatrix} \frac{\partial m}{\partial x_1}(\mathbf{x}^*(t_f), t_f) \\ \frac{\partial m}{\partial x_2}(\mathbf{x}^*(t_f), t_f) \\ \frac{\partial m}{\partial t}(\mathbf{x}^*(t_f), t_f) \end{bmatrix} \triangleq \begin{bmatrix} \frac{\partial m}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) \\ \hline \frac{\partial m}{\partial t}(\mathbf{x}^*(t_f), t_f) \end{bmatrix} \tag{5.1-51}$$

in the three-dimensional space. Thus, admissible variations must be normal to the vector (5.1-51), so

$$\left[\frac{\partial m}{\partial x_1}(\mathbf{x}^*(t_f), t_f)\right] \delta x_{1_f} + \left[\frac{\partial m}{\partial x_2}(\mathbf{x}^*(t_f), t_f)\right] \delta x_{2_f} + \left[\frac{\partial m}{\partial t}(\mathbf{x}^*(t_f), t_f)\right] \delta t_f = 0. \tag{5.1-52}$$

For the surface specified we have

$$2[x_1^*(t_f) - 3]\,\delta x_{1_f} + 2[x_2^*(t_f) - 4 - t_f]\,\delta x_{2_f} - 2[x_2^*(t_f) - 4 - t_f]\,\delta t_f = 0. \tag{5.1-53}$$

Solving for δt_f gives

$$\delta t_f = \frac{[x_1^*(t_f) - 3]}{[x_2^*(t_f) - 4 - t_f]}\,\delta x_{1_f} + \delta x_{2_f}. \tag{5.1-54}$$

Substituting in Eq. (5.1-18) and collecting terms, we obtain

$$\begin{aligned} &\left[\frac{\partial h}{\partial x_1}(\mathbf{x}^*(t_f), t_f) - p_1^*(t_f) + \left[\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f)\right.\right. \\ &\quad \left.\left. + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f)\right]\left[\frac{x_1^*(t_f) - 3}{x_2^*(t_f) - 4 - t_f}\right]\right] \delta x_{1_f} \\ &\quad + \left[\frac{\partial h}{\partial x_2}(\mathbf{x}^*(t_f), t_f) - p_2^*(t_f) + \mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f)\right. \\ &\quad \left. + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f)\right] \delta x_{2_f} = 0. \end{aligned} \tag{5.1-55}$$

Since there is one constraint involving the three variables $(\delta x_{1_f}, \delta x_{2_f}, \delta t_f)$, δx_{1_f} and δx_{2_f} can be varied independently; therefore, the coefficients of δx_{1_f} and δx_{2_f} must be zero. This gives two equations; the third equation is

$$m(\mathbf{x}^*(t_f), t_f) = 0. \tag{5.1-56}$$

In general, we may have $1 \leq k \leq n$ relationships

$$\mathbf{m}(\mathbf{x}(t), t) = \begin{bmatrix} m_1(\mathbf{x}(t), t) \\ \cdot \\ \cdot \\ \cdot \\ m_k(\mathbf{x}(t), t) \end{bmatrix} = \mathbf{0}, \tag{5.1-57}$$

which must be satisfied by the $(n + 1)$ variables $\mathbf{x}(t_f)$ and t_f. Reasoning as in the situation where $\mathbf{m}$ is not dependent on time, we deduce that the admissible values of the $(n + 1)$ vector

$$\begin{bmatrix} \delta \mathbf{x}_f \\ \hline \delta t_f \end{bmatrix}$$

are normal to each of the gradient vectors

$$\begin{bmatrix} \frac{\partial m_1}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) \\ \hline \frac{\partial m_1}{\partial t}(\mathbf{x}^*(t_f), t_f) \end{bmatrix}, \ldots, \begin{bmatrix} \frac{\partial m_k}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) \\ \hline \frac{\partial m_k}{\partial t}(\mathbf{x}^*(t_f), t_f) \end{bmatrix}, \tag{5.1-58}$$

which are assumed to be linearly independent. Writing Eq. (5.1-18) as

$$\begin{bmatrix} \frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f) \\ \hline \mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f)\ t_f) \end{bmatrix}^T \begin{bmatrix} \delta \mathbf{x}_f \\ \hline \delta t_f \end{bmatrix} = 0 \triangleq \mathbf{v}^T \begin{bmatrix} \delta \mathbf{x}_f \\ \hline \delta t_f \end{bmatrix} \tag{5.1-59}$$

and again using the result that $\mathbf{v}$ must be a linear combination of the gradient vectors in (5.1-58), we obtain

$$\mathbf{v} = d_1 \begin{bmatrix} \frac{\partial m_1}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) \\ \hline \frac{\partial m_1}{\partial t}(\mathbf{x}^*(t_f), t_f) \end{bmatrix} + \cdots + d_k \begin{bmatrix} \frac{\partial m_k}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) \\ \hline \frac{\partial m_k}{\partial t}(\mathbf{x}^*(t_f), t_f) \end{bmatrix}, \tag{5.1-60}$$

or

$$\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f) = d_1 \left[\frac{\partial m_1}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f)\right] + \cdots + d_k \left[\frac{\partial m_k}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f)\right]$$

and

$$\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = d_1 \left[\frac{\partial m_1}{\partial t}(\mathbf{x}^*(t_f), t_f)\right] + \cdots + d_k\left[\frac{\partial m_k}{\partial t}(\mathbf{x}^*(t_f), t_f)\right]. \tag{5.1-61}$$

Equations (5.1-61), the k equations

$$\mathbf{m}(\mathbf{x}^*(t_f), t_f) = \mathbf{0}, \tag{5.1-62}$$

and the n equations $\mathbf{x}^*(t_0) = \mathbf{x}_0$ comprise a set of $(2n + k + 1)$ equations in the $2n$ constants of integration, the variables $d_1, d_2, \ldots, d_k$, and t_f. It is left as an exercise for the reader to verify that (5.1-62) and (5.1-61) yield Eqs. (5.1-55) and (5.1-56).

The boundary conditions which we have discussed are summarized in Table 5-1. Of course, mixed situations can arise, but these can be handled by returning to Eq. (5.1-18) and applying the ideas introduced in the preceding discussion.

Although the boundary condition relationships may look foreboding, setting up the equations is not difficult; obtaining solutions is another matter. This should not surprise us, however, for we already suspect that numerical techniques are required to solve most problems of practical interest. Let us now illustrate the determination of the boundary-condition equations by considering several examples.

Example 5.1-1. The system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -x_2(t) + u(t) \end{aligned} \tag{5.1-63}$$

is to be controlled so that its control effort is conserved; that is, the performance measure

$$J(u) = \int_{t_0}^{t_f} \tfrac{1}{2}u^2(t)\, dt \tag{5.1-64}$$

is to be minimized. The admissible states and controls are not bounded. Find necessary conditions that must be satisfied for optimal control.

The first step is to form the Hamiltonian

$$\mathscr{H}(\mathbf{x}(t), u(t), \mathbf{p}(t)) = \tfrac{1}{2}u^2(t) + p_1(t)x_2(t) - p_2(t)x_2(t) + p_2(t)u(t). \tag{5.1-65}$$

From Eqs. (5.1-17b) and (5.1-17c) necessary conditions for optimality are

$$\begin{aligned}\dot{p}_1^*(t) &= -\frac{\partial \mathscr{H}}{\partial x_1} = 0 \\ \dot{p}_2^*(t) &= -\frac{\partial \mathscr{H}}{\partial x_2} = -p_1^*(t) + p_2^*(t),\end{aligned} \tag{5.1-66}$$

and

$$0 = \frac{\partial \mathscr{H}}{\partial u} = u^*(t) + p_2^*(t). \tag{5.1-67}$$

If Eq. (5.1-67) is solved for $u^*(t)$ and substituted into the state equations (5.1-63), we have

$$\begin{aligned}\dot{x}_1^*(t) &= x_2^*(t) \\ \dot{x}_2^*(t) &= -x_2^*(t) - p_2^*(t).\end{aligned} \tag{5.1-68}$$

Equations (5.1-68) and (5.1-66)—the state and costate equations—are a set of $2n$ linear first-order, homogeneous, constant-coefficient differential equations. Solving these equations gives

$$\begin{aligned}x_1^*(t) &= c_1 + c_2[1 - \epsilon^{-t}] + c_3[-t - \tfrac{1}{2}\epsilon^{-t} + \tfrac{1}{2}\epsilon^{t}] \\ &\qquad + c_4[1 - \tfrac{1}{2}\epsilon^{-t} - \tfrac{1}{2}\epsilon^{t}] \\ x_2^*(t) &= c_2\epsilon^{-t} + c_3[-1 + \tfrac{1}{2}\epsilon^{-t} + \tfrac{1}{2}\epsilon^{t}] + c_4[\tfrac{1}{2}\epsilon^{-t} - \tfrac{1}{2}\epsilon^{t}] \\ p_1^*(t) &= c_3 \\ p_2^*(t) &= c_3[1 - \epsilon^{t}] + c_4\epsilon^{t}.\end{aligned} \tag{5.1-69}$$

Now let us consider several possible sets of boundary conditions.

a. Suppose $\mathbf{x}(0) = \mathbf{0}$ and $\mathbf{x}(2) = [5 \quad 2]^T$. From $\mathbf{x}(0) = \mathbf{0}$ we obtain $c_1 = c_2 = 0$; the remaining two equations to be solved are

$$\begin{aligned}5 &= c_3[-2 - \tfrac{1}{2}\epsilon^{-2} + \tfrac{1}{2}\epsilon^{2}] + c_4[1 - \tfrac{1}{2}\epsilon^{-2} - \tfrac{1}{2}\epsilon^{2}] \\ 2 &= c_3[-1 + \tfrac{1}{2}\epsilon^{-2} + \tfrac{1}{2}\epsilon^{2}] + c_4[\tfrac{1}{2}\epsilon^{-2} - \tfrac{1}{2}\epsilon^{2}].\end{aligned} \tag{5.1-70}$$

Solving these linear algebraic equations gives $c_3 = -7.289$ and $c_4 = -6.103$, so the optimal trajectory is

$$\begin{aligned}x_1^*(t) &= 7.289t - 6.103 + 6.696\epsilon^{-t} - 0.593\epsilon^{t} \\ x_2^*(t) &= 7.289 - 6.696\epsilon^{-t} - 0.593\epsilon^{t}.\end{aligned} \tag{5.1-71}$$

b. Let $\mathbf{x}(0) = \mathbf{0}$ and $\mathbf{x}(2)$ be unspecified; consider the performance measure

$$J(u) = \tfrac{1}{2}[x_1(2) - 5]^2 + \tfrac{1}{2}[x_2(2) - 2]^2 + \tfrac{1}{2}\int_0^2 u^2(t)\,dt. \tag{5.1-72}$$

Table 5-1 SUMMARY OF BOUNDARY CONDITIONS IN OPTIMAL CONTROL PROBLEMS

Problem	*Description*	*Substitution in Eq. (5.1-18)*	*Boundary-condition equations*	*Remarks*
t_f fixed	1. $\mathbf{x}(t_f) = \mathbf{x}_f$ specified final state	$\delta\mathbf{x}_f = \delta\mathbf{x}(t_f) = \mathbf{0}$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \mathbf{x}_f$	$2n$ equations to determine $2n$ constants of integration
	2. $\mathbf{x}(t_f)$ free	$\delta\mathbf{x}_f = \delta\mathbf{x}(t_f)$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) - \mathbf{p}^*(t_f) = \mathbf{0}$	$2n$ equations to determine $2n$ constants of integration
	3. $\mathbf{x}(t_f)$ on the surface $\mathbf{m}(\mathbf{x}(t)) = \mathbf{0}$	$\delta\mathbf{x}_f = \delta\mathbf{x}(t_f)$ $\delta t_f = 0$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f)) - \mathbf{p}^*(t_f) = \sum_{i=1}^{k} d_i\left[\frac{\partial m_i}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right]$ $\mathbf{m}(\mathbf{x}^*(t_f)) = \mathbf{0}$	$(2n + k)$ equations to determine the $2n$ constants of integration and the variables $d_1, \ldots, d_k$
t_f free	4. $\mathbf{x}(t_f) = \mathbf{x}_f$ specified final state	$\delta\mathbf{x}_f = \mathbf{0}$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \mathbf{x}_f$ $\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = 0$	$(2n + 1)$ equations to determine the $2n$ constants of integration and t_f
	5. $\mathbf{x}(t_f)$ free		$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f) = \mathbf{0}$ $\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = 0$	$(2n + 1)$ equations to determine the $2n$ constants of integration and t_f
	6. $\mathbf{x}(t_f)$ on the moving point $\boldsymbol{\theta}(t)$	$\delta\mathbf{x}_f = \left[\frac{d\boldsymbol{\theta}}{dt}(t_f)\right]\delta t_f$	$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\mathbf{x}^*(t_f) = \boldsymbol{\theta}(t_f)$ $\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f)$ $+ \left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f)\right]^T \left[\frac{d\boldsymbol{\theta}}{dt}(t_f)\right] = 0$	$(2n + 1)$ equations to determine the $2n$ constants of integration and t_f

7. $\mathbf{x}(t_f)$ on the surface $\mathbf{m}(\mathbf{x}(t)) = \mathbf{0}$		$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f) = \sum_{i=1}^{k} d_i\left[\frac{\partial m_i}{\partial \mathbf{x}}(\mathbf{x}^*(t_f))\right]$ $\mathbf{m}(\mathbf{x}^*(t_f)) = \mathbf{0}$ $\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f) = 0$	$(2n + k + 1)$ equations to determine the $2n$ constants of integration, the variables $d_1, \ldots, d_k$, and t_f
8. $\mathbf{x}(t_f)$ on the moving surface $\mathbf{m}(\mathbf{x}(t), t) = \mathbf{0}$		$\mathbf{x}^*(t_0) = \mathbf{x}_0$ $\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f) - \mathbf{p}^*(t_f) = \sum_{i=1}^{k} d_i\left[\frac{\partial m_i}{\partial \mathbf{x}}(\mathbf{x}^*(t_f), t_f)\right]$ $\mathbf{m}(\mathbf{x}^*(t_f), t_f) = \mathbf{0}$ $\mathscr{H}(\mathbf{x}^*(t_f), \mathbf{u}^*(t_f), \mathbf{p}^*(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}^*(t_f), t_f)$ $= \sum_{i=1}^{k} d_i\left[\frac{\partial m_i}{\partial t}(\mathbf{x}^*(t_f), t_f)\right]$	$(2n + k + 1)$ equations to determine the $2n$ constants of integration, the variables $d_1, \ldots, d_k$, and t_f.

The modified performance measure affects only the boundary conditions at $t = 2$. From entry 2 of Table 5-1 we have

$$\begin{aligned} p_1^*(2) &= x_1^*(2) - 5 \\ p_2^*(2) &= x_2^*(2) - 2. \end{aligned} \tag{5.1-73}$$

c_1 and c_2 are again zero because $\mathbf{x}^*(0) = \mathbf{0}$. Putting $t = 2$ in Eq. (5.1-69) and substituting in (5.1-73), we obtain the linear algebraic equations

$$\begin{bmatrix} 0.627 & -2.762 \\ 9.151 & -11.016 \end{bmatrix} \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}. \tag{5.1-74}$$

Solving these equations, we find that $c_3 = -2.697$, $c_4 = -2.422$; hence,

$$\begin{aligned} x_1^*(t) &= 2.697t - 2.422 + 2.560\epsilon^{-t} - 0.137\epsilon^{t} \\ x_2^*(t) &= 2.697 - 2.560\epsilon^{-t} - 0.137\epsilon^{t}. \end{aligned} \tag{5.1-75}$$

c. Next, suppose that the system is to be transferred from $\mathbf{x}(0) = \mathbf{0}$ to the line

$$x_1(t) + 5x_2(t) = 15 \tag{5.1-76}$$

while the original performance measure (5.1-64) is minimized. As before, the solution of the state and costate equations is given by Eq. (5.1-69), and $c_1 = c_2 = 0$. The boundary conditions at $t = 2$ are, from entry 3 of Table 5-1,

$$\begin{aligned} x_1^*(2) + 5x_2^*(2) &= 15 \\ -p_1^*(2) &= d \\ -p_2^*(2) &= 5d. \end{aligned} \tag{5.1-77}$$

Eliminating d and substituting $t = 2$ in (5.1-69), we obtain the equations

$$\begin{bmatrix} 15.437 & -20.897 \\ 11.389 & -7.389 \end{bmatrix} \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 15 \\ 0 \end{bmatrix}, \tag{5.1-78}$$

which have the solution $c_3 = -0.894$, $c_4 = -1.379$. The optimal trajectory is then

$$\begin{aligned} x_1^*(t) &= 0.894t - 1.379 + 1.136\epsilon^{-t} + 0.242\epsilon^{t} \\ x_2^*(t) &= 0.894 - 1.136\epsilon^{-t} + 0.242\epsilon^{t}. \end{aligned} \tag{5.1-79}$$

Example 5.1-2. The space vehicle shown in Fig. 5-4 is in the gravity field of the moon. Assume that the motion is planar, that aerodynamic forces are negligible, and that the thrust magnitude T is constant. The control