Direct Collocation

Tuomas Raivio
Raimo Ehtamo
Raimo P. Hämäläinen

Abstract

A discretization scheme for optimal control methods, collocation, is described. It replaces the original infinite dimensional problem with a finite-dimensional approximation and allows the use of ordinary nonlinear optimization. The method seems to produce rapidly results that are accurate enough for most purposes.

Consider an optimal control problem of the Mayer form, hereafter referred to as P1,

$$\begin{aligned} \min \psi(x(T), T)& \\ \text{subject to}& \\ \dot{x}(t) &= f(x(t), u(t)) \\ x(0) &= x_{init} \\ x(T) &= x_{final} \\ C(x(t), u(t)) &\leq 0 \\ S(x(t)) &\leq 0, \;\; t \in [0, T] \end{aligned}$$

where $x(t) \in R^n$, $u(t) \in R^u$, $f : R^n \times R^u \mapsto R^n$, $C : R^n \times R^u \mapsto R^c$ and $S : R^n \mapsto R^s$. Possible explicit time dependence of $f(\cdot)$ may be suppressed with a new independent variable and problems of Bolza type, i.e. with integral cost functional, can be turned into Mayer form by adding a new state variable. The final time T may be fixed or free.

In the method of direct collocation, the finite dimensional solution subspace is the space of piecewise polynomials of time and given degree, defined in the interval $t \in [0, T]$. We use Hermite interpolation with 3rd degree polynomials for the state variables and linear polynomials for the control variables. The state equation must be satisfied in the middle of each interval.

For simplicity, consider an equidistant division of the solution interval

$$t_j = j\frac{T}{m} := j\Delta t, \;\; j = 0, \ldots, m.$$

In the jth subinterval we seek state component trajectories of the form

$$x_{ij}(t) = a + bt + ct^2 + dt^3, \;\; t \in [t_{j-1}, t_j]. \tag{1}$$

We hereafter drop the subscripts i and j for clarity.Introducing a new transformed time variable

$$\tau := \frac{t - t_{j-1}}{\Delta t}$$

and differentiating expression (1) with respect to τ yields the following system of equations:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} x(0) \\ \dot{x}(0) \\ x(1) \\ \dot{x}(1) \end{pmatrix}.$$

The independent variable is τ and ($\dot{}$) means differentiation with respect to τ. Evaluating (1) at $\tau = 1/2$ and substituting the coefficients solved from the above system of equations leads to

$$x(1/2) = \frac{x(0) + x(1)}{2} + \Delta t \frac{f(0) - f(1)}{8},$$

where $f(\tau)$ is an abbreviation of $f_i(x(\tau), u(\tau))$ and refers to the corresponding state equation component. Note that $\frac{dx}{d\tau} = \Delta t \frac{dx}{dt}$. In the same way we obtain the expression for $\dot{x}(1/2)$:

$$\dot{x}(1/2) = -3 \frac{x(0) - x(1)}{2\Delta t} - \frac{f(0) + f(1)}{4}.$$

Using the expression for $x(1/2)$ and linear interpolation of the controls, $f(1/2)$ may be calculated. Define the defect at the center of the interval j as

$$\Delta_j := \dot{x}(1/2) - f(1/2).$$

When the values of the state variables at the ends of the interval are chosen such that the defect is driven to zero, the cubic provides an approximation of the state component trajectory without explicit integration. The controls at the time points may now be selected freely within their bounds to minimize the objective function, as far as the constraints $\Delta_j = 0$, initial and terminal constraints, and possible state constraints are satisfied. Thus the infinite dimensional optimal control problem P1 may be approximated by an ordinary finite dimensional nonlinear optimization problem

$$\begin{array}{rcl}
\min\limits_{(x_0,x_1,\ldots,x_m,u_0,u_1,\ldots,u_m;T)} & & \psi(x_m,T) \\
\text{subject to} & & \\
\Delta_j & = & 0,\ \ j=1,\ldots,m \\
x_0 & = & x_{init} \\
x_m & = & x_{final} \\
S(x_j) & \leq & 0,\ \ j=0,\ldots,m \\
C(x_j,u_j) & \leq & 0,\ \ j=0,\ldots,m \\
-T & \leq & 0.
\end{array}$$

Here x_j refers to state vector x at the time instant t_j. The state constraints may be satisifed only pointwise, since the differential equations are satisfied only in the middle points of the segments. If violations occur, the time division should be made denser to suppress them.

Applying direct collocation leads to a nonlinear optimization problem where the number of the decision variables is $(n+u)(m+1)+1$ when the final time is free. The number of constraints amounts to $nm+(n_{cieq}+n_{sieq})(m+2)+n_{init}+n_{final}$, where n_{cieq}, n_{sieq} refer to the number of control and state inequality constraints and n_{init} and n_{final} to the number of initial and final conditions, respectively. The nonlinearity of the collocation constraints depends on the state equations. Some of the state and control variable constraints may be simple bounds.