Analysis

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August 23rd, 2022

This note summarizes some of the key concepts from the first set of lectures.

Euclidean Space

In order to make much progress with any kind of economic modeling or analysis, we need to find a space that is both rich enough to capture many aspects of the problem but structured enough that analysis of the model is possible. For most practical purposes, this space will be euclidean space, $\mathbb{R}^n$. We equip this space with a notion of distance called a *norm*. For any $x \in \mathbb{R}^n$:

$$||x|| := \sqrt{\sum_{i=1}^{n} x_i^2}$$

or in vector notation $\sqrt{x^T x}$.[1] We use this norm to define three special kind of sets:

Definition. *An* Open Ball/Neighborhood *of x with radius ϵ is the set $B_\epsilon(x) = \{x \in \mathbb{R}^n : ||x|| < \epsilon\}$.*

Definition. *A set $O \subseteq \mathbb{R}^n$ is an* open set *if for every $x \in O$, there exists an $\epsilon > 0$ s.t. $B_\epsilon(x) \subseteq O$.*

Definition. *A set $C \subseteq \mathbb{R}^n$ is a* closed set *if $C^c = \mathbb{R}^n \setminus C$ is open. Equivalently, a set is closed if it contains all of its limit points.*[2]

Economic "applications" of open sets/norms/continuity (beyond the extreme value theorem)

1. Having an idea of closeness allows us to begin to ask questions like how much does "demand for pizza change in response to small changes in the price of cheese?"
2. Lets us think about the "stability" of things. A typical macro question: If an economy is a certain steady state, will it return to that steady state after a small, unexpected, temporary change.
3. (More abstract:) Lets us think about the "robustness" of models or assumptions. If I make a small change to some assumptions, does it lead to only a small change in behavior.
4. Econometrics: What happens to an estimator in the limit as the sample size goes to infinity?

Sequences, Continuity and Compactness

A *sequence* $(x_n)_{n=1}^{\infty}$ is a countably infinite ordered list of numbers. x_n converges to x if for any $\epsilon > 0$ there exists an N s.t. if $n \geq N$ then $||x_n - x|| < \epsilon$.[3] Open sets allow us to make small changes without leaving the set, closed sets allow us to move along sequences. This gives us a natural way to definite *continuity*, e.g. functions whose value doesn't respond much to small changes.

[1] The norm satisfies

1. $||x|| \geq 0$, with equality iff $x = 0$.
2. $||ax|| = |a|||x||$ for any $a \in \mathbb{R}$.
3. $||x + y|| \leq ||x|| + ||y||$ (triangle inequality)
4. $(x \cdot y)^2 \leq ||x||||y||$ (Cauchy-Schwarz inequality)

[2] x is a limit point of C if for every $\epsilon > 0$, $B_\epsilon(x) \cap C$ contains a $y \neq x$.

[3] Example: $x_n = \frac{1}{n}$ converges to 0. $x_n = n$ diverges to ∞. $x_n = (-1)^n$ does not converge or diverge.

Definition. $f : X \to Y$ *is* continuous *at* $x \in X$ *if for any* $\epsilon > 0$ *there exists a* $\delta > 0$ *s.t.* $||f(x) - f(y)|| < \epsilon$ *whenever* $||x - y|| < \delta$.

A function is *continuous* if it is continuous at every point in the domain.[4]

A set $K \subseteq \mathbb{R}^n$ is *compact* if every sequence in K has a convergent subsequence.[5] This leads us to our two major results. First, we characterize compactness:

Theorem 1 (Bolzano–Weierstrass theorem)**.** *A set is compact if and only if it is closed and bounded.*

This gives us a simpler condition to check for compactness. Which is good, because it's the key ingredient in our main theorem

Theorem 2 (Extreme Value Theorem)**.** *If* X *is compact and* $f : X \to \mathbb{R}$ *is continuous then* $f(x)$ *has a maximum.*

[4] TFAE:

1. $f : X \to Y$ is continuous.
2. For any open set O, $f^{-1}(O)$ is open.
3. For any convergent sequence $(x_n)_{i=1}^n$ with limit x, $f(x_n) \to f(x)$.

[5] You may see this defined on wikipedia or in other sources as "every open cover has a finite subcover." This is equivalent to the definition we're using (which is usually called sequential compactness) in Euclidean space.

Some places where we could use the extreme Value theorem:

1. Consumer theory: Demand exists. Given a vector of prices, there is a bundle of goods that the consumer prefers over all other affordable bundles.
2. Producer theory: The firm can find a production plan that maximizes it's profits.
3. Econometrics: Establishing extremum estimators are well defined, e.g. least squares, maximum likelihood estimation, minimum distance estimation etc.

Non-exhaustive list of notation.

- $\mathbb{N}$: Natural numbers, e.g. 1,2,5, sometimes 0.
- $\mathbb{R}$: Real numbers. $\mathbb{R}^n$ is space of N-dimensional real valued vectors, $\mathbb{R}^{m\times n}$ is the space of $m \times n$ matricies.
- Vectors: denoted by lower case letters e.g. $x, y, z, a, b, c \in \mathbb{R}^M$. Treated as column vectors unless otherwise stated. x_i is ith component, i, j, k, l in denote indicies of components.[6]
- $x \leq y$: I use this to mean $x_i \leq y_i$ for all i. Similarly $x = y$ is $x_i = y_i$ for all i. [7]
- Dot/Inner product: $x \cdot y$ or $\langle x, y \rangle$.
- Matricies: denoted by upper case letters, typically early in the alphabet e.g. $A, B, C \in \mathbb{R}^{n\times m}$.[8]
- Sets: denoted by upper case letters, typically either near the end of the alphabet X, Y, Z. X^c is the complement, $X \cap Y$ is the intersection, $X \cup Y$ is the union, $P(X)$ or 2^X is the power set. $A \subseteq B$ denotes that A is a subset of B
- Functions: denoted by lower case letters, typically letters near f in the alphabet e.g. f, g, h. $f : X \to Y$ is the function that maps each element of X to an element of Y.[9]
- Correspondences: Typically denoted with upper case (usually greek) letters.
- $||x||$: The norm of vector x.
- $\frac{df}{dx}$: The derivative of f. Sometimes f'.
- $\frac{\partial f}{\partial x}$: The partial derivative of f with respect to x. Sometimes f_x.
- ∇f: The gradient of f.
- $D^2 f$: The hessian (second derivative matrix) of f. Sometimes Hf.
- max / min: The max and min operator. The largest/smallest value in a set.[10]

[6] Endogenous variables will usually be letters at the end of the alphabet, exogenous will be letters at the beginning or lower case greek letters. 0 denotes the vector of all 0's with the correct dimension. Similar convention for matricies.

[7] There's not really standard notation for this. Your first year textbooks/notes may use a different convention, e.g. $x \leq y$ also requires $x \neq y$, so be careful. For any $x, y \in \mathbb{R}^n$ $x \cdot y = x^T y = \sum_{i=1}^N x_i y_i$.

[8] AB is the product of matrix A and B. $Det(A)$ is the determinant. X^T is the transpose. $(X)_{ij}$ is the $i \times j$th element.

[9] $f(X)$ is the set $\{y : y = f(x) \text{ for some } x \in X\}$. f^{-1} is the inverse of f (sometimes a function, sometimes not). $f \circ g$ is f composed with g.

[10] $\max_{x\in X} f(x)$ is the largest value in the set $f(X)$.

- sup / inf: The supremum and infimum operator. The smallest upper bound/greatest lower bound of a set.[11]
- Sequences: Denoted by vectors with a subscript e.g. x_n, y_m. The whole sequence is written as $(x_n)_{n=1}^{\infty}$, subsequences have additional subscripts e.g. x_{n_k}.
- Limits: $\lim_{n\to\infty} x_n$ or $\lim_{x\to y} f(x)$. $\limsup x_n$ is the lim superior, e.g. $\lim_{n\to\infty} \sup_{k\geq n} x_k$.
- Random Variables: Capital letters, generally near the end of the alphabet, e.g. X, Y, Z.
- Probability: P or Pr.
- Expected value $E(X)$.

[11] Unlike the maximum, these always exist for subsets of $\mathbb{R}$ (or are $\pm\infty$).