Aalto University
School of Electrical
Engineering

# Lecture 5: Open-Loop Dynamics of a DC Motor

## ELEC-E8405 Electric Drives

Marko Hinkkanen

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# Learning Outcomes

After this lecture and exercises you will be able to:

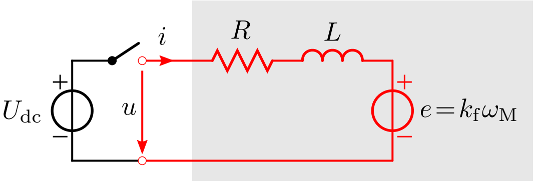
- Draw relevant block diagrams of the DC motor
- Derive transfer functions based on the block diagram
- Interpret the most essential properties of second-order systems
- Explain the concept of time-scale separation

# Introduction

- Open-loop (plant) model of the DC motor
  - Combination of the electrical and mechanical models
  - Plant model is the starting point in the control design
- Brief recap on control theory tools in the context of the DC motor
  - Block diagram, transfer function, 2nd-order system, state-variable form
  - Basic knowledge of these tools is needed in the field of electric drives (and in many other fields as well)
- Transient response in open loop (speed and current)
- Time-scale separation (electrical and mechanical subsystems)

Note: Controllers will not be considered today

# Example: Connection of a DC Voltage Source to the Terminals




- Assume that a DC voltage source is connected to the motor terminals
- How will the speed $\omega_{\mathrm{M}}$ and the current $i$ behave?
- How to model and analyse transient response in more general cases?

# Outline

# DC Motor Model

- Voltage equation

$$L\frac{\mathrm{d}i}{\mathrm{d}t} = u - Ri - e$$

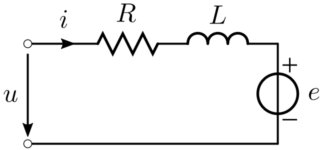
where $e = k_\mathrm{f}\omega_\mathrm{M}$ is the back emf

- Motion equation

$$J\frac{\mathrm{d}\omega_\mathrm{M}}{\mathrm{d}t} = \tau_\mathrm{M} - \tau_\mathrm{L}$$

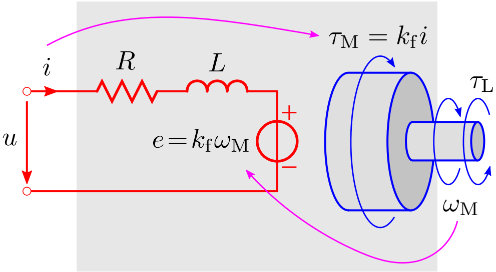
where $\tau_\mathrm{M} = k_\mathrm{f} i$ is the electromagnetic torque

- For simplicity, the flux factor $k_\mathrm{f}$ is assumed to be constant in the following



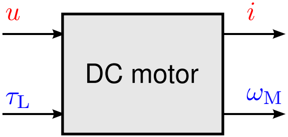
# Electrical and Mechanical Dynamics Are Coupled

Physical model




$\Rightarrow$

Block diagram



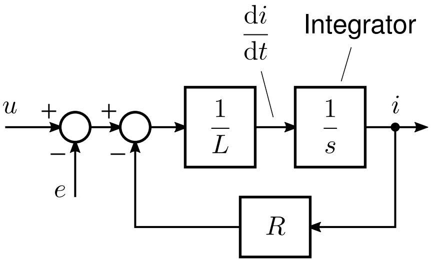
# Electrical Dynamics in the Time Domain

- Differential equation

$$L\frac{\mathrm{d}i}{\mathrm{d}t} = u - e - Ri$$

- $u$ and $e$ are the inputs
- $i$ is the output
- Integration of both sides gives

$$i = \int \frac{1}{L}\,(u - e - Ri)\,\mathrm{d}t$$




- In the time domain, $s = \mathrm{d}/\mathrm{d}t$ refers to the differential operator

In some textbooks, the symbol $p = \mathrm{d}/\mathrm{d}t$ is used for the differential operator in the time domain.

# Electrical Dynamics in the Laplace Domain

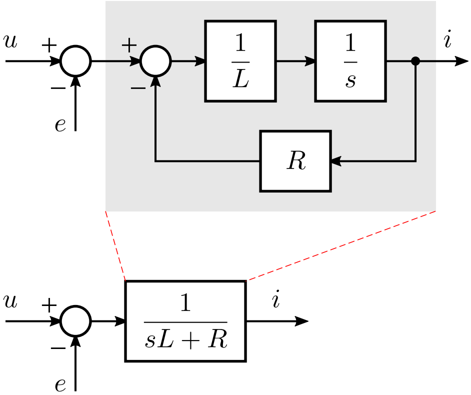
- Laplace transform: $\mathrm{d}/\mathrm{d}t \to s$
- Current can be solved

$$i(s) = \frac{1}{sL + R}[u(s) - e(s)]$$

- Transfer function (admittance)

$$Y(s) = \frac{1}{sL + R} = \frac{1/R}{1 + Ts}$$

where $T = L/R$

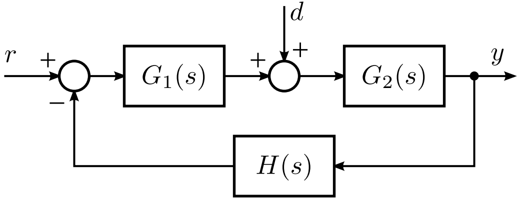



In the Laplace domain, $s = \sigma + \mathrm{j}\omega$ is a complex variable. However, the differential operator and the Laplace variable can be used interchangeably in many cases.

# Useful Block Diagram Algebra

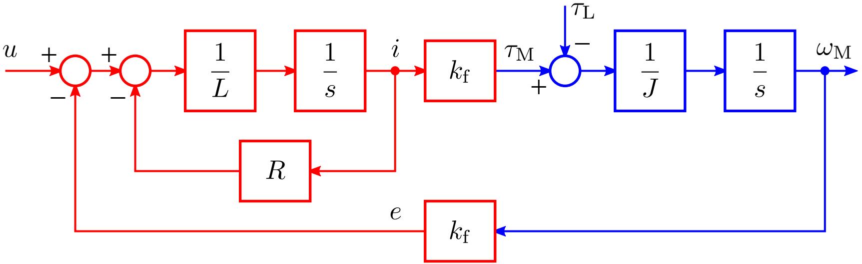
$$\frac{y(s)}{r(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}$$

$$\frac{y(s)}{d(s)} = \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}$$



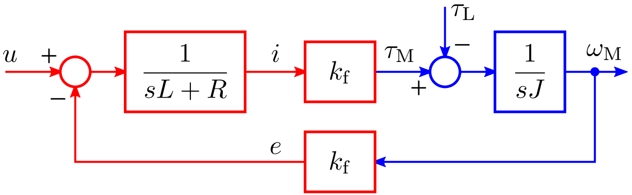
# Block Diagram of the DC Motor

$$L\frac{\mathrm{d}i}{\mathrm{d}t} = -Ri - k_\mathrm{f}\omega_\mathrm{M} + u$$
$$J\frac{\mathrm{d}\omega_\mathrm{M}}{\mathrm{d}t} = k_\mathrm{f} i - \tau_\mathrm{L}$$




- Flux factor $k_\mathrm{f}$ couples the electrical and mechanical dynamics

# Block Diagram of the DC Motor




- Armature current depends on the armature voltage and the load torque

$$i(s) = G_{iu}(s)u(s) + G_{i\tau}(s)\tau_L(s)$$

- Speed depends on the armature voltage and the load torque

$$\omega_M(s) = G_{\omega u}(s)u(s) + G_{\omega\tau}(s)\tau_L(s)$$

- Could you derive the transfer functions based on the block diagram?

# Transfer Function From $u(s)$ to $\omega_{\mathrm{M}}(s)$

- Transfer function from the voltage $u(s)$ to the speed $\omega_{\mathrm{M}}(s)$

$$G_{\omega u}(s) = \frac{\dfrac{k_{\mathrm{f}}}{JL}}{s^2 + \dfrac{R}{L}s + \dfrac{k_{\mathrm{f}}^2}{JL}} = \frac{K\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

- Last form is a typical generic form of 2nd-order systems
- Undamped angular frequency, damping ratio, and DC gain

$$\omega_0 = \frac{k_{\mathrm{f}}}{\sqrt{JL}} \qquad \zeta = \frac{R}{2k_{\mathrm{f}}}\sqrt{\frac{J}{L}} \qquad K = \frac{1}{k_{\mathrm{f}}}$$

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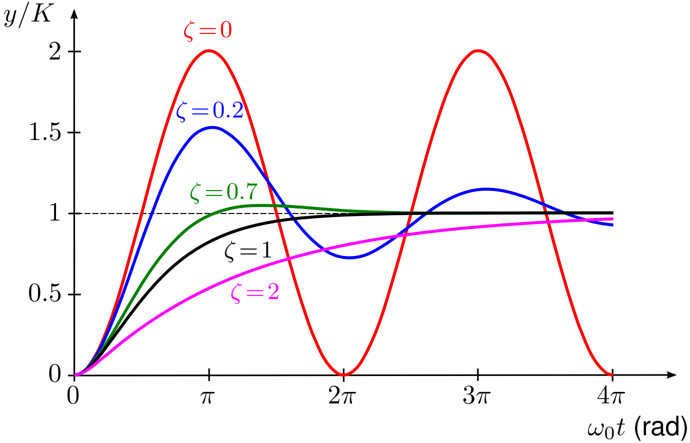
You don't need to remember these more complex transfer functions, but practise deriving them based on the block diagram instead. However, you should remember the generic form used above.

# 2nd-Order System in the Time Domain: Step Response

- 2nd-order system

$$G(s) = \frac{y(s)}{u(s)} = \frac{K\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

- Response $y(t)$ to the step input $u(t)$ is shown
- No overshoot if $\zeta \geq 1$




Step responses can be easily plotted using numerical simulation tools. If needed, an analytical solution could be obtained using the inverse Laplace transformation.

# 2nd-Order System in the Frequency Domain

- 2nd-order system

$$G(s) = \frac{K\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

- Consider a sinusoidal input

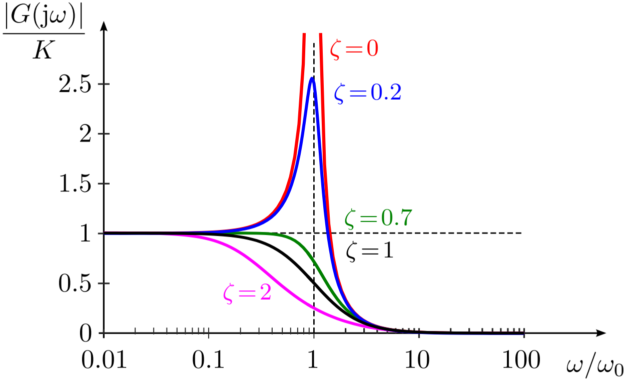
$$u(t) = U\sin(\omega t)$$

- For $\zeta > 0$, the output in steady state is

$$y(t) = AU\sin(\omega t + \phi)$$

where

$$A = |G(\mathrm{j}\omega)| \qquad \phi = \angle G(\mathrm{j}\omega)$$



# Transfer Function From $u(s)$ to $i(s)$

- Transfer function from the voltage $u(s)$ to the current $i(s)$

$$G_{iu}(s) = \frac{s/L}{s^2 + \frac{R}{L}s + \frac{k_\mathrm{f}^2}{JL}}$$

- Characteristic polynomial remains the same
(holds also for other transfer functions of the system)
- Zero at $s = 0$ in this transfer function
- If $J \to \infty$ (i.e. $\omega_\mathrm{M}$ is constant)

$$G_{iu}(s) = \frac{1}{sL + R} = Y(s)$$

# State-Variable Form

- State-variable model consists of coupled 1st-order differential equations
- Derivatives $\mathrm{d}\boldsymbol{x}/\mathrm{d}t$ depend on the states $\boldsymbol{x}$ and the system input $u$

$$
\begin{aligned}
\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u \\
y &= \boldsymbol{C}\boldsymbol{x}
\end{aligned}
$$

- States $x$ depend on the history, but not on the present values of the inputs
- Output $y$ depends only on the states (in physical systems)
- State variables are typically associated with the energy storage
  - Current $i$ of an inductor (or its flux linkage $\psi = Li$)
  - Voltage $u$ of a capacitor (or its charge $q = Cu$)
  - Speed $v$ of a mass (or its momentum $p = mv$)
- Choice of state variables is not unique (as shown in the parenthesis above)

# State-Variable Form of the DC Motor

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} i \\ \omega_\mathrm{M} \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} -\frac{R}{L} & -\frac{k_\mathrm{f}}{L} \\ \frac{k_\mathrm{f}}{J} & 0 \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} i \\ \omega_\mathrm{M} \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}}_{\boldsymbol{B}_u} u + \underbrace{\begin{bmatrix} 0 \\ -\frac{1}{J} \end{bmatrix}}_{\boldsymbol{B}_\tau} \tau_\mathrm{L}$$

$$i = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\boldsymbol{C}_i} \boldsymbol{x} \qquad \omega_\mathrm{M} = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{\boldsymbol{C}_\omega} \boldsymbol{x}$$

- Transfer function from $u(s)$ to $\omega_\mathrm{M}(s)$ as an example

$$G_{\omega u}(s) = \boldsymbol{C}_\omega (s\boldsymbol{I} - \boldsymbol{A})^{-1} \boldsymbol{B}_u$$

- Transfer functions of the system are unique, i.e. the state-variable form leads to the previous transfer functions
- Poles of the transfer function are eigenvalues of the system matrix $\boldsymbol{A}$

# Outline

# Time-Domain Simulation Examples

Rated values of a small PM DC motor

- Voltage $u_{\mathrm{N}} = 110$ V
- Current $i_{\mathrm{N}} = 10$ A
- Rotation speed $n_{\mathrm{N}} = 1200$ r/min

- Angular speed

$$\begin{aligned}\omega_{\mathrm{N}} &= 2\pi n_{\mathrm{N}} \\ &= 2\pi \cdot \frac{1200 \text{ r/min}}{60 \text{ s/min}} \\ &= 125.7 \text{ rad/s}\end{aligned}$$

Electrical parameters

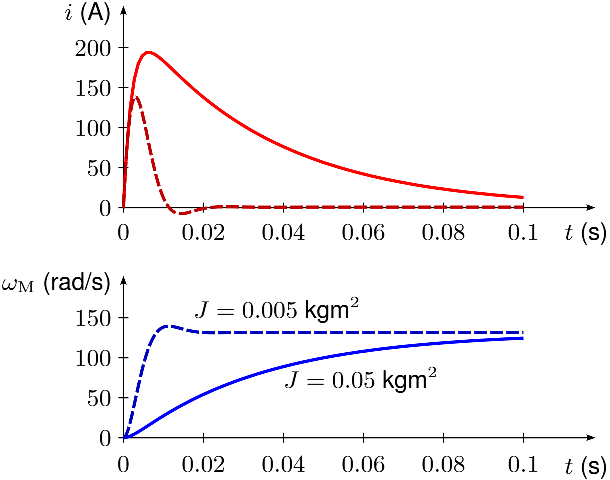
- $R = 0.5\ \Omega$
- $L = 1$ mH
- $k_{\mathrm{f}} = 0.836$ Vs

Two inertia values

- Case 1: $J = 0.05$ kgm$^2$ ($\zeta = 2.11$, $\omega_0 = 118$ rad/s)
- Case 2: $J = 0.005$ kgm$^2$ ($\zeta = 0.67$, $\omega_0 = 374$ rad/s)

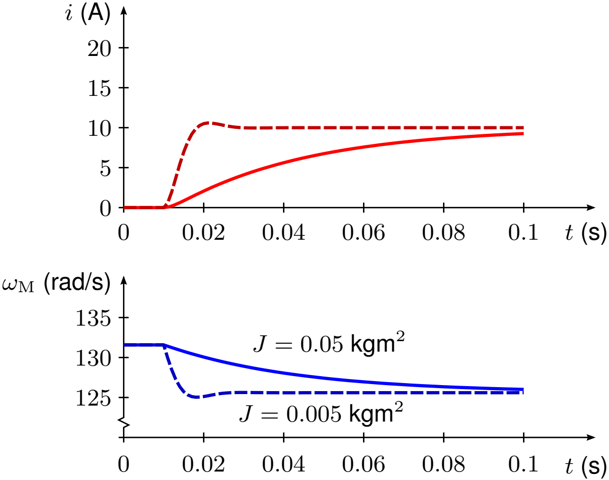
# Voltage-Step Response

- Terminals are connected to the rated voltage
- Load torque is zero
- Current rises quickly and then decreases as the back-emf $e = k_\mathrm{f}\omega_\mathrm{M}$ increases
- Very large current peak is undesirable



# Load-Torque-Step Response

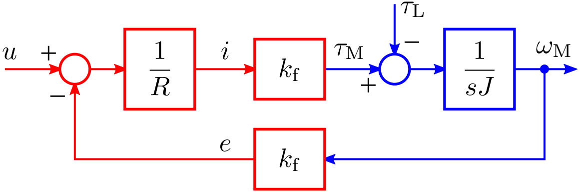
- Armature voltage is constant (rated)
- Initially no-load condition
- Rated load torque is applied at $t = 0.01$ s



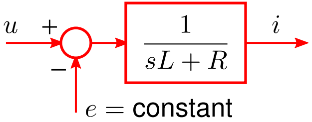
# Outline

# Time-Scale Separation

- When considering the slow mechanical dynamics, the quickly converging electrical dynamics may be approximated with the DC gain

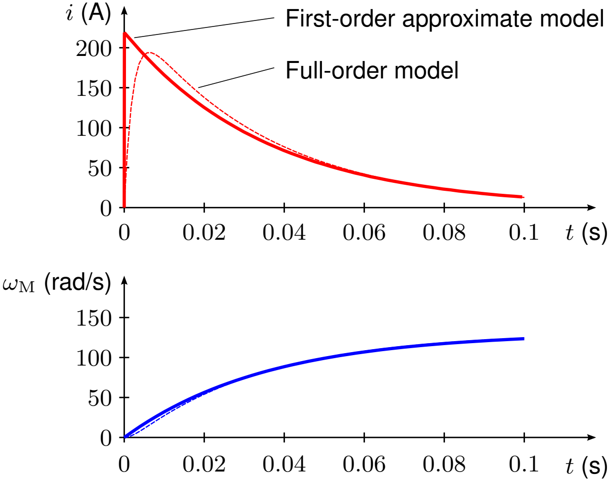



- When considering the fast electrical dynamics, the slowly varying rotor speed may be assumed to be constant



# Reduced-Order Model for Slow Mechanical Dynamics

- Response to the rated voltage step
- Electrical dynamics are approximated with the steady-state gain
- Response of the reduced-order model is close to the full-order model



# Reduced-Order Model for Fast Electrical Dynamics

- Response to the rated voltage step
- Speed is assumed to be constant
- Fast electrical transient is well modelled using the first-order model $Y(s)$
- Notice a different scale of the time axes compared to the previous case