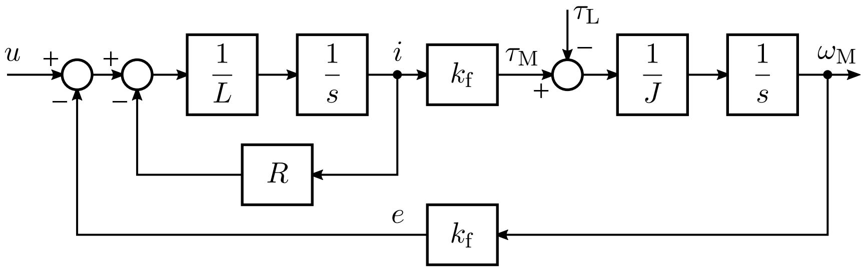
Problem 1: Transfer functions of a DC motor

The block diagram of a DC motor is shown in the figure.

(a) Derive the transfer functions

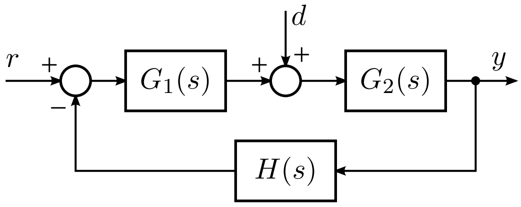
$$G_{\omega u}(s) = \frac{\omega_\mathrm{M}(s)}{u(s)} \quad \text{and} \quad G_{\omega\tau}(s) = \frac{\omega_\mathrm{M}(s)}{\tau_\mathrm{L}(s)}$$

(b) Replace the electric dynamics of the machine with the DC gain and formulate the transfer functions $G_{\omega u}(s)$ and $G_{\omega\tau}(s)$.



Solution

(a) Consider a closed-loop system shown in the figure.

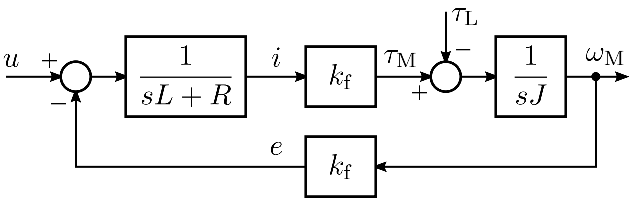


The following equations hold for the closed-loop transfer functions:

$$\frac{y(s)}{r(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)} \tag{1}$$

$$\frac{y(s)}{d(s)} = \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)} \tag{2}$$

It is relatively easy to derive these equations if one has forgotten them. Using (1), the block diagram given in the problem is first transformed to the following form:



Using (1), we can write the transfer function from the voltage to the speed as

$$G_{\omega u}(s) = \frac{\omega_\mathrm{M}(s)}{u(s)} = \frac{\frac{1}{sL+R}k_\mathrm{f}\frac{1}{sJ}}{1 + \frac{1}{sL+R}k_\mathrm{f}\frac{1}{sJ}k_\mathrm{f}} = \frac{\frac{k_\mathrm{f}}{JL}}{s^2 + s\frac{R}{L} + \frac{k_\mathrm{f}^2}{JL}}$$

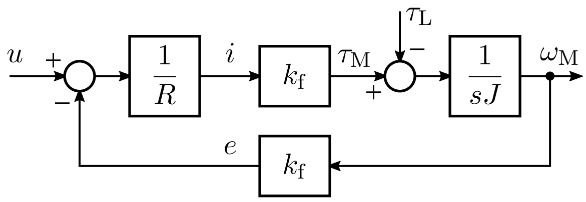
Using (2), the transfer function from the load torque to the speed becomes

$$G_{\omega\tau}(s) = \frac{\omega_\mathrm{M}(s)}{\tau_\mathrm{L}(s)} = -\frac{\frac{1}{sJ}}{1 + \frac{1}{sL+R}k_\mathrm{f}\frac{1}{sJ}k_\mathrm{f}} = -\frac{\frac{1}{J}\left(s + \frac{R}{L}\right)}{s^2 + s\frac{R}{L} + \frac{k_\mathrm{f}^2}{JL}}$$

(b) The electric dynamics of the machine

$$Y(s) = \frac{1}{sL + R}$$

is replaced with the DC gain by substituting $s = 0$. The corresponding block is shown in the following figure:



The transfer functions can be derived from this block diagram in a fashion similar to Part (a) of the problem. The same result is obtained by multiplying the numerator and denominator of the derived transfer functions by L and then substituting $L = 0$:

$$G_{\omega u}(s) = \frac{\omega_\mathrm{M}(s)}{u(s)} = \frac{\frac{k_\mathrm{f}}{JR}}{s + \frac{k_\mathrm{f}^2}{JR}}$$

$$G_{\omega\tau}(s) = \frac{\omega_\mathrm{M}(s)}{\tau_\mathrm{L}(s)} = -\frac{1/J}{s + \frac{k_\mathrm{f}^2}{JR}}$$

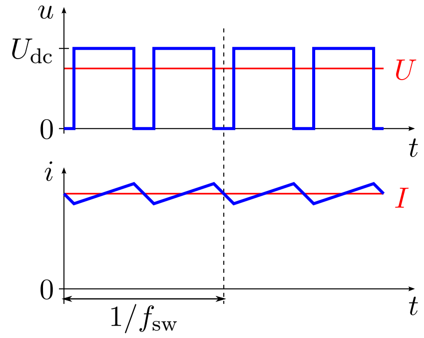
The time constant of the first-order system is

$$T = \frac{JR}{k_\mathrm{f}^2}$$

which depends strongly on the flux factor k_f of the motor (and increases when the flux factor is decreased).

Problem 2: Current ripple

The parameters of a DC motor are: $R = 1\ \Omega$, $L = 10$ mH, and $k_\mathrm{f} = 4$ Vs. The average steady-state current taken by the motor is $I = 100$ A and the rotor speed is 560 r/min. The motor is supplied from a four-quadrant DC-DC converter, where the unipolar PWM is applied. The DC-bus voltage is $U_\mathrm{dc} = 450$ V and the switching (carrier) frequency is $f_\mathrm{sw} = 4$ kHz. Calculate the peak-to-peak current ripple.



Solution

The electrical dynamics of the DC motor are governed by

$$L\frac{\mathrm{d}i(t)}{\mathrm{d}t} = u(t) - Ri(t) - e(t) \tag{3}$$

The rotor angular speed is

$$\omega_{\mathrm{M}} = 2\pi \cdot \frac{560 \text{ r/min}}{60 \text{ s/min}} = 58.6 \text{ rad/s}$$

and the steady-state back-emf is $E = k_{\mathrm{f}}\omega_{\mathrm{M}} = 4 \text{ Vs} \cdot 58.6 \text{ rad/s} = 234.6 \text{ V}$. Based on (3), the average voltage in the steady state is

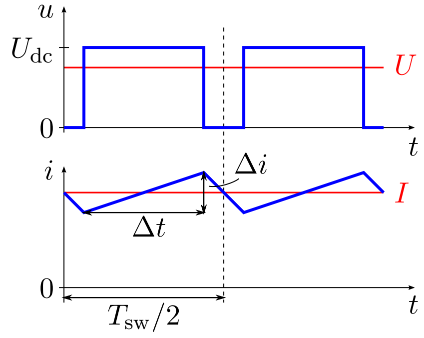
$$U = RI + E = 1\ \Omega \cdot 100 \text{ A} + 234.6 \text{ V} = 334.6 \text{ V}$$

The figure below illustrates the waveforms of the voltage u and the current i. The average voltage during the switching period T_{sw} is

$$\overline{u} = \frac{1}{T_{\mathrm{sw}}}\int_0^{T_{\mathrm{sw}}} u(t)\mathrm{d}t = \frac{2\Delta t}{T_{\mathrm{sw}}}U_{\mathrm{dc}} \tag{4}$$

where Δt is the duration of the positive voltage pulse (see the figure). Since $U = \overline{u}$ in the steady state, the duration is

$$\Delta t = \frac{U}{U_{\mathrm{dc}}}\frac{T_{\mathrm{sw}}}{2} = \frac{U}{U_{\mathrm{dc}}}\frac{1}{2f_{\mathrm{sw}}}$$



In the time scale of switching periods, the dynamics in (3) can be approximated as

$$L\frac{\mathrm{d}i(t)}{\mathrm{d}t} = u(t) - RI - E = u(t) - U \tag{5}$$

The change Δi in the current during the positive voltage pulse $u(t) = U_{\mathrm{dc}}$ is

$$\begin{aligned}\Delta i &= \frac{U_{\mathrm{dc}} - U}{L}\Delta t \\ &= \frac{U_{\mathrm{dc}} - U}{L}\frac{U}{U_{\mathrm{dc}}}\frac{1}{2f_{\mathrm{sw}}} \\ &= \frac{450\ \mathrm{V} - 334.6\ \mathrm{V}}{10\ \mathrm{mH}} \cdot \frac{334.6\ \mathrm{V}}{450\ \mathrm{V}}\frac{1}{2 \cdot 4\ \mathrm{kHz}} = 1.1\ \mathrm{A}\end{aligned}$$

This peak-to-peak current ripple is roughly 1% of the average current.

Remark 1: Naturally, the same result would be obtained, if the zero-voltage condition $u(t) = 0$ were used:

$$\Delta i = \frac{U}{L}\left(\frac{T_{\mathrm{sw}}}{2} - \Delta t\right)$$

Remark 2: The switching period is $T_{\mathrm{sw}} = 1/f_{\mathrm{sw}} = 1/(4\ \mathrm{kHz}) = 250\ \mu\mathrm{s}$ and the electrical time constant is $T = L/R = 10$ ms. Since T is much longer (40 times) than T_{sw}, the approximation (5) holds well.