First some theory:

- The present value $P_x(r)$ of a cash flow $x = (x_0, x_1, \ldots, x_n)$ at an annual interest rate r is:

$$P_x(r) = x_0 + \frac{x_1}{1+r} + \frac{x_1}{(1+r)^2} + \cdots + \frac{x_n}{(1+r)^n} = \sum_{i=0}^{n} \frac{x_i}{(1+r)^i}.$$

  Defining discount factor $d = 1/(1+r)$ yields

$$P_x(d) = x_0 + dx_1 + d^2x_2 \cdots + d^nx_n = \sum_{i=0}^{n} d^i x_i.$$

- If the interest is compounded more frequently than annually, the interest rate and discount factor must be adjusted correspondingly. For example, monthly compounding at a yearly interest rate of r means that an interest rate of $r_m = r/12$ is applied every month. Similarly, the monthly discount factor is $d_m = 1/(1 + r_m) = 1/(1 + r/12)$.

- The internal rate of return (IRR) of a cash flow $x = (x_0, x_1, \ldots, x_n)$ is the interest rate r that satisfies

$$P_x(r) = x_0 + \frac{x_1}{1+r} + \ldots \frac{x_n}{(1+r)^n} = 0.$$

  That is, IRR is that interest rate at which the present value of a cash flow is 0. The equation does not have an analytic solution in general, but it is almost always easy to solve the the equation numerically. For example, function IRR(values;guess) of Excel calculates IRR for the cash flow given as an argument.

- Inflation means an increase in general prices with time. Assuming a constant annual inflation rate $f = 2\%$, 1000 € today corresponds to 1020 € next year in terms of purchasing power. Had this capital been invested in a bank account that pays 5% interest, it would have grown to 1050 €. However, taking inflation into account yields an increased purchasing power of 1050 €/1.02 = 1029 €. Thus in terms of purchasing power, the *real interest rate* is only 2.9%.

1.1 (L2.5) A major European lottery advertises that it pays the winner 10 million euros. However, this prize money is paid at the rate of 500 000 € each year (with the first payment being immediate) for a total of 20 payments. What is the present value of this prize at 10% interest?

Solution:
Interest rate $r = 0.10$ gives a discount factor $d = 1/(1 + 0.10) \approx 0.909$.
Cash flow of the price is $x = (x_0, x_1, \ldots, x_{19})$, where $x_i = 500\ 000\, e$ for all $i = 0, \ldots, 19$.
Present value of the cash flow is $P_x(d) = x_0 + dx_1 + \cdots + d^{19}x_{19} \approx \underline{4\ 682\ 460\, e}$.

1.2 (L2.6) A young couple has made a nonrefundable deposit of the first month's rent (equal to 700 €) on a 6-month apartment lease. The next day they find a different apartment they like just as well, but its monthly rent is only 600 €. Assume an interest rate of 12%.
a) Should they switch to the new apartment, if they plan to be in the apartment only 6 months?
b) What if they plan to stay 1 year?

Solution:
a) Alternative cash flows:
Old apartment: $x = (x_0, x_1, \ldots, x_5)$, where $x_i = -700\, e$ for all $i = 0, \ldots, 5$.
New apartment: $y = (y_0, y_1, \ldots, y_5)$, where $y_0 = -700\, e - 600\, e = -1300\, e$ and $y_i = -600\, e$ for all $i = 1, \ldots, 5$.
Monthly interest rate is $r_m = 0.12/12 = 0.01$ and the monthly discount factor is $d_m = 1/(1+0.01) \approx 0.990$.

Present values of the cash flows
$P_x(d_m) = x_0 + d_m x_1 + \cdots + d_m^5 x_5 \approx -4097\, e$.
$P_y(d_m) = y_0 + d_m y_1 + \cdots + d_m^5 y_5 \approx -4212\, e$.
$\Rightarrow P_x(d_m) > P_y(d_m)$, i.e., the new apartment will be more costly. $\Rightarrow$ <u>Apartment should **not** be changed.</u>

b) Alternative cash flows
Old apartment: $x = (x_0, x_1, \ldots, x_{11})$, where $x_i = -700\, e$ for all $i = 0, \ldots, 11$.
New apartment: $y = (y_0, y_1, \ldots, y_{11})$, where $y_0 = -700\, e - 600\, e = -1300\, e$ and $y_i = -600e$ for all $i = 1, \ldots, 11$.
Monthly interest rate is $r_m = 0.12/12 = 0.01$ and the monthly discount factor is $d_m = 1/(1+0.01) \approx 0.990$.

Present values of the cash flows
$P_x(d_m) = x_0 + d_m x_1 + \cdots + d_m^{11} x_{11} \approx -7957\, e$.
$P_y(d_m) = y_0 + d_m y_1 + \cdots + d_m^{11} y_{11} \approx -7521\, e$.
$\Rightarrow P_x(d_m) < P_y(d_m)$, i.e., the old apartment will be more costly.$\Rightarrow$ <u>Apartment should be changed.</u>

1.3 (L2.8) Two copy machines are available. Both have useful lives of 5 years. One machine can either be leased or purchased outright; the other must be purchased. Hence there are a total of three options: A (leasing machine 1), B (purchasing machine 1) and C (purchasing machine 2). The details are shown in Table 1. (The first year's maintenance is included in the initial cost. There are then four additional maintenance payments, occurring at the beginning of each year, followed by revenues from resale.) The present values of the expenses of these three options using a 10% interest rate are also indicated in the table. According to a present value analysis, the machine of least cost, as measured by the present value, should be selected; that is, option B.

It is not possible to compute the IRR for any of these alternatives, because all cash flows are not positive. However, it is possible to calculate the IRR on an incremental basis (that is, the IRR with which a change from an alternative to another has a zero present value). If this exceeds the prevailing interest rate, a change can be justified.
a) Find the IRR corresponding to a change from A to B. Is a change from A to B justified on the basis of the IRR?
b) What about a change from B to C?

Table 1: Copy Machine Options

	Alternative		
	A	**B**	**C**
Initial outlay	-6000	-30000	-35000
Yearly expense (paid at the start of the year)	-8000	-2000	-1600
Resale value (paid at the end of the year)	0	10000	12000
Present value (@10%)	-39359	-32131	-34221

Solution:
The cash flows are
Alternative A: $x = (x_0, \dots, x_5)$, where $x_0 = -14000\,e$, $x_i = -8000e$ for all $i = 1, \dots, 4$ and $x_5 = 0\,e$.
Alternative B: $y = (y_0, \dots, y_5)$, where $y_0 = -32000\,e$, $y_i = -2000\,e$ for all $i = 1, \dots, 4$ ja $y_5 = 10000\,e$.
Alternative C: $z = (z_0, \dots, z_5)$, where $z_0 = -36600\,e$, $z_i = -1600\,e$ for all $i = 1, \dots, 4$ ja $z_5 = 12000\,e$.

At interest rate $r = 0.10$, the present values of the cash flows are
Alternative A: $P_x(r) = \sum_{i=0}^{5} \frac{1}{(1+r)^i} x_i \approx -39359\,e$.
Alternative B: $P_y(r) = \sum_{i=0}^{5} \frac{1}{(1+r)^i} y_i \approx -32131\,e$.
Alternative C: $P_z(r) = \sum_{i=0}^{5} \frac{1}{(1+r)^i} z_i \approx -34221\,e$.

Incremental cash flows for both changes in alternatives:

a) A change from alternative A to alternative B (B-A). Cash flow $a = (a_0, \ldots, a_5) = (y_0 - x_0, \ldots, y_5 - x_5)$, where $a_0 = -32000 - (-14000) = -18000$, $a_i = -2000 - (-8000) = 6000$ for all $i = 1, \ldots, 4$ and $a_5 = 10000 - 0 = 10000$. The present value of cash flow a is $P_a(r) = \sum_{i=0}^{5} a_i/(1+r)^i \approx 7228\,e$. Moreover, the IRR of a can be calculated (e.g., with Excel) to be $r_a \approx 0.24$, and because $r_a = 0.24 > 0.10 = r$, a change from alternative A to alternative B should be made.

b) A change from alternative B to alternative C (C-B). Cash flow $b = (b_0, \ldots, b_5) = (z_0 - y_0, \ldots, z_5 - y_5)$, where $b_0 = -36600 - (-32000) = -4600$, $b_i = -1600 - (-2000) = 400$ for all $i = 1, \ldots, 4$ and $b_5 = 12000 - 10000 = 2000$. The present value of cash flow b is $P_b(r) = \sum_{i=0}^{5} b_i/(1+r)^i \approx -2090\,e$. Moreover, the IRR of a can be calculated (e.g., with Excel) to be $r_b \approx -0.06$, and because $r_b = -0.06 < 0.10 = r$, a change from alternative B to alternative C should **not** be made.

1.4 (L2.13) In general, we say that two projects with cash flows $x = (x_0, x_1, \ldots, x_n)$ and $y = (y_0, y_1, \ldots, y_n)$ *cross*, if $x_0 < y_0$ and $\sum_{i=0}^{n} x_i > \sum_{i=0}^{n} y_i$. Let $P_x(d)$ and $P_y(d)$ denote the present values of these two projects as a function of the discount factor $d = 1/(1+r)$.
a) Show that there is a crossover calue $c > 0$ such that $P_x(c) = P_y(c)$.
b) For cash flows is Table 2, calculate the crossover value c.

Table 2: Cash flows of projects 1 and 2

	Year					
	0	**1**	**2**	**3**	**4**	**5**
Project 1	-100	30	30	30	30	30
Project 2	-150	42	42	42	42	42

Solution
a) By definition, $x_0 - y_0 < 0$ and $\sum_{i=0}^{n} x_i - \sum_{i=0}^{n} y_i > 0 \Leftrightarrow (\sum_{i=0}^{n} x_i - y_i) > 0$. We define a cash flow $z = (z_0, \ldots, z_n)$ so that $z_i = x_i - y_i$ for all $i = 0, \ldots, n$. Consequently, $z_0 < 0$ and $\sum_{i=0}^{n} z_i > 0$. Thus, for the present value $P_z(d) = \sum_{i=0}^{n} d^i z_i$ of cash flow z we have $P_z(0) < 0$ and $P_z(1) > 0$, and therefore based on the Intermediate Value Theorem (Bolzano Theorem) there exists $c \in [0, 1]$ such that $P_z(c) = 0$, which completes the proof.

b) The cash flows of projects 1 and 2 are $x = (-100, 30, 30, 30, 30, 30)$ and $y = (-150, 42, 42, 42, 42, 42)$. Thus, we have $x_0 = -100 > -150 = y_0$ and $\sum_{i=0}^{5} x_i = 50 < 60 = \sum_{i=0}^{5} y_i$. Hence according to part a) of this exercise, there exists $c \in [0, 1]$ such that $P_z(c) = 0$, where z is defined as $z = (y_0 - x_0, \ldots, y_5 - x_5)$. Computing the intersection with Excel yields $c \approx 0.9398$.

1.5 (L2.15) A division of ABBOX Corporation has developed the concept of a new product. Production of the product would require initial capital expenditure of 10 million euros. It is anticipated that 1 million units would be sold each year for 5 years, and then the product would be obsolete and production would cease. Each year's production would require 10,000 hours of labor and 100 tons of raw material. Currently (year 0) the average wage rate is 30 € per hour and the cost of the raw material is 100 € per ton. The product would sell for 3.30 € per unit, and this price is expected to be maintained (in real terms). ABBOX management likes to use a 12% discount rate for projects of this type and faces a 34% tax rate on profit. The initial capital expenditure can be depreciated in a straight-line fashion over 5 years. (Depreciation reduces the taxable profit, but is not otherwise included in the investment calculation.)
a) In its first analysis of this project, management did not apply inflation factors to the extrapolated revenues and operating costs. What present value did they obtain?
b) How would the answer change if an inflation rate of 4% was applied? Inflation increases wage rate, raw material costs and the price of the product.

Solution:
Initial information:

Initial investment:	$I = 10,000,000$ (€)
Anticipated demand:	$S = 1,000,000$ (units/year)
Production requirements:	$p_l = 10,000$ (hours of labour/year) and
	$p_m = 100$ (raw material tons/year)
Initial wage rate:	$c_l = 30$ (€/ hour of labour)
Initial cost of raw material:	$c_m = 100$ (€/raw material ton)
Initial price for the product:	$h = 3.30$ (€/ unit)
Interest rate:	$r = 0.12$
Tax rate:	$v = 0.34$
Annual depreciation:	$P = 2000000$ (€)

Revenue = Product demand · Product price= $Sh = 1000000 \cdot 3.30 = 3300000$ (€/ year).
Costs = Required hours of labour · Wage rate + Required raw material · Raw material cost
$= p_l c_l + p_m c_m = 10000 \cdot 30 + 100 \cdot 100 = 310000$ (€/ year).
Profit = Revenue - Costs = 2990000 (€/ year).

a) The following table presents the calculations of the profit (or loss) after taxes after each period.

	Year					
	0	**1**	**2**	**3**	**4**	**5**
Gross Profit	0	2990000	2990000	2990000	2990000	2990000
Depreciation	0	-2000000	-2000000	-2000000	-2000000	-2000000
Pretax Income	0	990000	990000	990000	990000	990000
Taxes	0	-336600	-336600	-336600	-336600	-336600
Profit after taxes	0	2653400	2653400	2653400	2653400	2653400

The cash flow of the first year (0) is the initial investment. The cash flow of the following years is the profit after taxes. Hence the present value of the investment can be calculated as:

	Year						
	0	**1**	**2**	**3**	**4**	**5**	Σ
Cash flow	-10000000	2653400	2653400	2653400	2653400	2653400	3267000
Present value	-10000000	2369107	2115274	1888638	1686284	1505610	-435087

Thus, the present value of the investment is $\underline{-435087\,e}$, when the inflation is not taken into account. Hence the investment **would not** be profitable according to these calculations.

b) When inflation is taken into account, the price of the product and the costs of raw material and labour increase according to the inflation rate $f = 0.04$. The annual profit is now:

Profit = Revenue$\cdot 1.04^{k-1}$-Costs$\cdot 1.04^{k-1}$=(Revenue-Costs)$\cdot 1.04^{k-1}$, where $k = 1, \ldots, 5$ is the year.

The exponent is $k-1$, because the accounting is made at the end of each year k. Hence the profit will increase from part a) of this exercise. A similar table as in part a) for the calculations of the profit (or loss) after taxes after each period is presented below.

	Year					
	0	**1**	**2**	**3**	**4**	**5**
Gross Profit	0	2990000	3109600	3233984	3363343	349777
Depreciation	0	-2000000	-2000000	-2000000	-2000000	-2000000
Pretax Income	0	990000	1109600	1233984	1363343	1497877
Taxes	0	-336600	-377264	-419555	-463537	-509278
Profit after taxes	0	2653400	2732336	2814429	2899807	2988599

The cash flow of the first year (0) is again the initial investment, and the cash flow of the following years is the profit after taxes, which leads to the following calculation of the present value of the investment:

	Year						
	0	**1**	**2**	**3**	**4**	**5**	Σ
Cash flow	-10000000	2653400	2732336	2814429	2899807	2988599	
Present value	-10000000	2369107	2178202	2003255	1842880	1695811	89254

Thus, the inflation-adjusted present value of the investment is $\underline{89254\,e}$. Hence the investment **is** profitable according to the calculations with inflation taken into account.

Note the treatment of depreciation. Even though the 10 million investment is made at the beginning of the cycle of 5 year, it is depreciated from the taxable income evenly during the operating period. Would the investment be accounted only in the first calendar year, the first year would have severe losses and the tax rate of the following years would be excessive. Division of the depreciation into five years allows for much fairer taxation.