- Perpetual annuity pays a fixed sum periodically *forever*. Suppose an amount A is paid at the end of each period, and suppose the per-period interest rate is r. Then the present value of the perpetual annuity is
$P = \sum_{k=1}^{\infty} \frac{A}{(1+r)^k} = \frac{A}{1+r} + \sum_{k=2}^{\infty} \frac{A}{(1+r)^k} = \frac{A}{1+r} + \frac{1}{1+r} \sum_{k=1}^{\infty} \frac{A}{(1+r)^k} = \frac{A}{1+r} + \frac{P}{1+r}$
$\Rightarrow \underline{P = \frac{A}{r}}.$

- A finite-lifetime constant payment is termed annuity. Assume an amount A to be paid periodically n times, and assume a per-period interest rate r. The present value of the annuity is
$P = \sum_{k=1}^{n} \frac{A}{(1+r)^k} = \sum_{k=1}^{\infty} \frac{A}{(1+r)^k} - \sum_{k=n+1}^{\infty} \frac{A}{(1+r)^k} = \frac{A}{r} - \frac{1}{(1+r)^n} \sum_{k=1}^{\infty} \frac{A}{(1+r)^k} = \frac{A}{r} - \frac{1}{(1+r)^n} \frac{A}{r}$
$\Rightarrow \underline{P = \frac{A}{r}\left[1 - \frac{1}{(1+r)^n}\right]}$

- The amount A of the payment corresponding a present value P under periodical interest rate r and number of payment periods n can be derived from the above formula to be
$P = \frac{A}{r}\left[1 - \frac{1}{(1+r)^n}\right] \Rightarrow \underline{A = \frac{r(1+r)^n P}{(1+r)^n - 1}}.$

- The issuer of a bond has an obligation to pay the bond holder (according to the rules specified at the time the bond is issued)
a) the **face value** F of the bond at the date of maturity.
b) the possible **coupon payments** C/m periodically, for example a percentage (**coupon rate** c) of the face value, i.e., cF, where $c = C/(mF)$

- The present value of a bond is $P = \frac{F}{(1+\frac{\lambda}{m})^n} + \sum_{k=1}^{n} \frac{\frac{C}{m}}{(1+\frac{\lambda}{m})^k}$, where λ = the **yield** of the bond (i.e., the **yield to maturity**).

  Similarly to the case of the annuity formula, the collapsed form of the present value formula of a bond is
$P = \frac{F}{(1+\frac{\lambda}{m})^n} + \frac{\frac{C}{m}}{\frac{\lambda}{m}}\left[1 - \frac{1}{(1+\frac{\lambda}{m})^n}\right] = \frac{F}{(1+\frac{\lambda}{m})^n} + \frac{C}{\lambda}\left[1 - \frac{1}{(1+\frac{\lambda}{m})^n}\right].$

- The duration of a fixed-income security is a weighted average of the times that payments (cash flows) are made. For a zero-coupon bond, the duration equals the maturity, otherwise the duration is always less than the maturity. As the coupon rate increases, the duration decreases, because the cash flow has a greater weight in the earlier periods. The duration is formally defined as

$$D = \frac{PV(t_0)t_0 + PV(t_1)t_1 + \cdots + PV(t_n)t_n}{PV}, \text{ where } PV = \sum_{k=0}^{n} PV(t_k).$$

  For every non-negative cash flow, it is clear that $t_0 \leq D \leq t_n$.

- The calculation of the duration depends on the way the present value is calculated. If the present values of the bond are calculated using the yield of the bond (yield to maturity), then the formula becomes the Macaulay duration. Specifically, suppose a financial instrument makes m payments in a year, with a payment k being c_k (both coupon payment and possibly the face value), and there are n periods remaining. Then, the payment times are $t_k = k/m$ and the Macaulay duration can be calculated as

$$D = \frac{\sum_{k=1}^{n} \frac{k}{m} \frac{c_k}{(1+\frac{\lambda}{m})^k}}{PV}, \text{ where } PV = \sum_{k=1}^{n} \frac{c_k}{(1+\frac{\lambda}{m})^k}.$$

  If the coupon payments are identical ($c_k = C/m \;\forall\; k < n$ and $c_n = C/m + F$, where F is the face value of the bond and C the annual coupon payment), noting the coupon rate as $c = C/(mF)$, the explicit formula for the Macalay duration is

$$D = \frac{1+y}{my} - \frac{1+y+n(c-y)}{mc[(1+y)^n - 1] + my}, \text{ where } y = \frac{\lambda}{m}.$$

  The derivation of this formula can be found in the additional material for this exercise.

2.1 (L3.8) (Variable-rate mortgage) The Lehtinen family just took out a variable-rate mortgage on their new home. The mortgage value is 100 000 €, the term is 30 years, and initially the interest rate is 4%. The interest rate is guaranteed for 5 years, after which time the rate will be adjusted according to prevailing rates. The new rate can be applied to their loan either by changing the payment amount or by changing the length of the mortgage.
a) What is the original yearly mortgage payment? (Assume payments are yearly.)
b) What will be the mortgage balance after 5 years?
c) If the interest rate on the mortgage changes to 5% after 5 years, what will be the new yearly payment that keeps the termination time the same?
d) Under the interest change in c., what will be the new term if the payments remain the same?

Solution:

a) $P = 100\ 000$ €, $r = 0.04$
Payment time is 30 years, and because the payments are yearly, $n = 30$.

$$A = \frac{r(1+r)^n P}{(1+r)^n - 1} \approx \underline{5\ 783\ \text{€}}.$$

b) According to part a), annual payment of 5783 € suffices paying a debt of 100 000 € in 30 years. We calculate the present value of a loan paid with these payments in 25 years. $n = 25$, $A = 5\ 783$ €

$$P = \frac{A}{r}\left[1 - \frac{1}{(1+r)^n}\right] \approx 90\ 343\ \text{€}.$$

This is the balance of the mortgage after 5 years. These 25 years correspond the years 6-30 in the 30-year payment program. Because an amount of 90 343 € can be amortized during these years, in years 1-5 an amount of $\underline{100\ 000 - 90\ 343 = 9\ 657\ \text{€}}$ will be amortized.

For sake of clarity, Table 1 below presents the principal of the loan, the payments, and the portions of the payments allocated to the interest and principal payments.

Table 1: Loan amortization during the first 5 years of the payment program.

Year	Principal P	Payment A	Interest payment rP	Principal payment $A - rP$
1	100000	5783.01	4000	1783.01
2	98216.99	5783.01	3928.68	1854.33
3	96362.66	5783.01	3854.51	1928.50
4	94434.16	5783.01	3777.37	2005.64
5	92428.51	5783.01	3697.14	2085.87
$\sum$				**9657.36**

c) According to part b) there is 90 343 € loan remaining after 5 years. Because the paid interest is greater, the yearly payment has to be increased in order to keep the termination time the same.

$P = 90\ 343$ €, $r' = 0.05$

Payment time 25 years with yearly payments $\to n = 25$. The new yearly payment is

$$A = \frac{r'(1+r')^n P}{(1+r')^n - 1} \approx \underline{6410\ €}.$$

Hence the yearly payment increases by $6\ 410 - 5\ 783 = 627$ €.

d) According to part b) there are 90 343 € loan remaining after 5 years. Because the paid interest is greater, the payment time has to be increased to keep the yearly payment the same.

$P = 90\ 343$ €, $r' = 0.05$, $A = 5\ 783$ €

To solve the loan payoff time, we solve the n for which $A = \dfrac{r'(1+r')^n P}{(1+r')^n - 1}$ as follows:

$$A = \frac{r'P}{1-(1+r')^{-n}} \Leftrightarrow \frac{A - r'P}{A} = \big(1+r'\big)^{-n}$$

$$\Leftrightarrow \ln\big(A - r'P\big) - \ln A = -n\ln(1+r')$$

$$\Leftrightarrow n = \frac{\ln A - \ln(A - r'P)}{\ln(1+r')} \approx 31.1 \text{ years.}$$

The total life of the mortgage is $5 + 31.1 = \underline{36.1 \text{ years}}$.

In exercises 2.2 and 2.3 the coupon payments are made every 6 months ($m = 2$). The face values F of the bonds are 100 scores and the annual coupon payment is C is the nominal value×coupon rate. In formula (3.3) of the course book notations $c = C/(mF)$ and $y = \lambda/m$ are used.

2.2 (L3.10) (Duration) Find the price and duration of a 10-year, 8% bond that is trading at a yield of 10%.

Solution:
Par value $F = 100$, coupon rate 8%, yearly coupon payments $C = 8$ €.
There are $m = 2$ yearly payments, hence the per-period coupon rate is $c = 8\%/2 = 4\%$
The bond matures in 10 years, therefore $n = 10m = 10 \cdot 2 = 20$.
The yearly yield is $\lambda = 10\%$, and the per-period yield is $y = \lambda/m = 10\%/2 = 5\%$.

First let us calculate the price and duration of the bond as

$$P = \frac{F}{(1+\lambda/m)^n} + \sum_{k=1}^{n} \frac{C/m}{(1+\lambda/m)^k}$$

$$D = \frac{1}{P} \sum_{k=1}^{n} \frac{k}{m} \frac{c_k}{(1+\lambda/m)^k},$$

where $c_k = C/m$, if $k < n$ and $c_n = C/m + F$. The calculations can be made with a spreadsheet software as presented in Table ??. In Table ??, discount factors refer to factors $1/(1+\lambda)^k$, and duration factors refer to factors $(k/m)/(1+\lambda/m)^k$. The duration term refers to duration factors multiplied by the cash flow of period k.

Table 2: Spreadsheet calcaulation of present value and duration

	Σ					
Period k		1	2	···	19	20
Cash flow c_k		4	4	···	4	104
Time k/m		0.5	1.0	···	9.5	10.0
Discount factor		0.95	0.91	···	0.4	0.38
Duration factor		0.48	0.91	···	3.76	3.77
PV of cash flow	**87.54**	3.81	3.63	···	1.58	39.20
Duration term	598.79	1.90	3.63	···	15.04	391.97
Duration	**6.84**					

The calculations can also be made using the collapsed formulas for the bond price and Macaulay duration. Use of the bond price formula gives $P = \frac{F}{(1+\frac{\lambda}{m})^n} + \frac{C}{\lambda}\Big[1 - \frac{1}{(1+\frac{\lambda}{m})^n}\Big] \approx \underline{87.54 \text{ €}}$,

and use of the Macaulay duration formula gives $D = \frac{1+y}{my} - \frac{1+y+n(c-y)}{mc[(1+y)^n - 1] + my} \approx \underline{6.84}$,
where $y = \lambda/m$

2.3 (L3.14) (Duration limit) Show that the limiting value of duration as maturity is increased to infinity is

$$D \to \frac{1+\frac{\lambda}{m}}{\lambda}.$$

For the bonds in Table (3.6) of the course book (where $\lambda = 0.05$ ja $m = 2$) we obtain $D \to 20.5$. Note that for large λ this limiting value approaches $1/m$, and hence the duration for large yields tends to be relatively short.

Solution:

$D = \dfrac{1+y}{my} - \dfrac{1+y+n(c-y)}{mc[(1+y)^n - 1] + my} \Leftrightarrow D = \dfrac{1+y}{my} - \dfrac{\frac{1+y}{n} + (c-y)}{mc[\frac{(1+y)^n}{n} - \frac{1}{n}] + \frac{my}{n}}$

When $n \to \infty$, $\dfrac{1+y}{n} \to 0$, $\dfrac{1}{n} \to 0$ and $\dfrac{my}{n} \to 0$, but $\dfrac{(1+y)^n}{n} \to \infty$. Hence $D \to \dfrac{1+y}{my} = \dfrac{1+\frac{\lambda}{m}}{\lambda}$.

2.4 (L3.6) (The biweekly mortgage) Here is a proposal that has been advanced as a way for homeowners to save thousands of dollars on mortgage payments: pay biweekly instead of monthly. Specifically, if monthly payments are x, it is suggested that one instead pays $x/2$ every two weeks (for a total of 26 payments per year). This will pay down the mortgage faster, saving interest. The savings are surprisingly dramatic for this seemingly minor modification - often cutting the total interest payment by over one-third. Assume a loan amount of 150 000 € for 20 years at 4% interest, compounded monthly.
a) Under a monthly payment program, what are the monthly payments and the total interest paid over the course of the 20 years?
b) Using the biweekly program, when will the loan be completely repaid, and what are the savings in total interest paid over the monthly program? (You may assume biweekly compounding for this part.)
c) Which factors do the savings consist of? Is this a newly invented money-making-machine? Compare the annual payments.

Solution:

a) $P = 150,000$ €, $r = 0.04, r_m = 0.04/12 \approx 0.0033$
Loan time is 20 years. Hence the total amount of monthly periods is: $n = 20 \times 12 = 240$
The monthly payments are: $P = \frac{A}{r}\left[1 - \frac{1}{(1+r)^n}\right] \Rightarrow A = \frac{r(1+r)^n P}{(1+r)^n - 1}$

$\Rightarrow A = \frac{0.0033(1.0033)^{240}150000}{(1.0033)^{240} - 1} \approx \underline{909\text{ €}}$.

The total payment is $nA \approx 218\ 153$ €.
The total interest paid is the total payments - the present value P of the loan (150 000 €):
The total interest paid is: $nA - P \approx \underline{68\ 153\text{ €}}$.

b) For bi-weekly payment there are 26 payments in a year. The bi-weekly payment is $A' = A/2 = 454$ €.
The per-period interest is $r_{26} = r/26 = 0.04/26 \approx 0.0015$
Using the formula from Exercise 2.1. for the loan payoff time we get
$n = \frac{\ln A' - \ln(A' - rP)}{\ln(1+r)} \approx 461,$
which is in years $461/26 \approx \underline{17.7}$.
Hence, the total payments are $nA' \approx 209\ 548$ €, from which the interest payments can be solved similarly to part a) as $nA - P \approx \underline{59\ 548\text{ €}}$.
Savings in total payments over monthly program is $218\ 153 - 209\ 548 = \underline{8\ 605\text{ €}}$ (12.6%). (We would get the same result when comparing the interest payments.)

c) The yearly payments are
part a): $12 \times 909 = 10\ 908$ €
part b): $26 \times 454 = 11\ 804$ €
Hence the annual payments are greater for the biweekly payment. For this reason it is natural that the interest payments are smaller.
If the total annual payment of part b) 11 804 € would be made monthly, the monthly payment would be 984 €, and using this payment the loan time and the interest payments would equal those of the bi-weekly program. Hence the savings result from faster payment program. Note that 1 mo. $\geq$ 4 wk. $= 2 \cdot 2$ wk.

2.5 A young couple is currently living on rent in an apartment with a monthly rent of 600 €. They have saved 10 000 € for buying an own apartment, and have consulted a real estate broker about the new apartment. The broker is offering them a two-room apartment with a price of 100 000 €, and with monthly costs of 120 € (consisting of maintenance and water payments). The apartment is in good condition and it requires no renovation. However, after three years the condominium will be renovating the windows of the apartment, causing a 2000 € payment at the start of the third year.

The couple has decided to take a 20-year annuity debt of 90 000 €, with an annual interest of 5% and with an opening payment of 100 €. However, because only 70% of the value of the apartment (70 000 €) can be used as collateral for the loan, the couple buys a partial collateral by the government for 15% of the value of the apartment (15 000 €) with a price of 375 € (2.5% of the size of the collateral), and the remaining 5000 € will be covered with a free collateral from the parents of the husband.
a) Considering only the payments during the time of the loan, should the couple move into the apartment? Note that effectively only 70% of the interest have to be paid, because 30% of the interests of the first apartment can be reduced from taxes.
b) What if only the payments to bank, government and the condominium are considered, and no amortizations are included in the calculations (amortizations increase the wealth of the couple)? Use the 5% interest rate for discounting.
c) What if the prevalent interest rate is 2.3% (12-month Euribor + 0.7 marginal)?

Solution:

Numerical values:

Rent now	= 600 €/month
Savings	= 10 000 €
Price of new apartment	= 100 000 €
Maintenance charge of the new apartment	= 120 €/month
Window renovation at the start of the third year	= 2000 €.
Loan payment time	= 20 years
Number of periods n	$= 20 \cdot 12 = 240$
Loan principal	= 90 000 €
Loan opening payment	= 100 €
Government partial collateral	= 375 €
Interest rate r	$= 0.05$
Monthly interest r_m	$= 0.05/12 = 0.004$

a) and b) The present value of the rent of the old apartment for the 20 years is $A' = 600$ €

$$P' = \frac{A'}{r_m}\left[1 - \frac{1}{(1+r_m)^n}\right] \approx 90\ 915 \text{ €}.$$

The monthly payments for the $P = 90\ 000$ € loan are

$$A = \frac{r_m(1+r_m)^n P}{(1+r_m)^n - 1} \approx 594 \text{ €}$$

The partial collateral and opening payments are paid immediately. The present values are 375 € and 100 €. The present value of the maintenance charges (using same formula as for the PV of the rent) is 18 183 €. The window renovation paid after three years (=36 months) has a present value of $2000/(1+r_m)^{36} \approx 1722$ €.

Let us consider the payments and their present values after the first period.

The monthly payment for the $P = 90\ 000$ € loan is 593.96 €.
The interest payments are $0.004 \cdot 90\ 000 = 375.00$ €.
The principal payment is $593.96 - 375 = 218.96$ €.
Because the effective interest payments are 70% of the total interest payments, the effective interest payments are $0.7 \cdot 375 = 262.50$ €.
The present values of the payments are calculated using the discount factor $d_m = 1/(1 + r_m)$. These calculations are concluded in Table **??** below.

Table 3: The payments after the first period and their present values

	Loan principal	Total payment	Interest payment	Principal payment	Eff. interest payment
	90 000	594	375.0	218.96	262.5
Present value	-	591.5	373.4	218.1	261.4

After the first period (and at the start of the second period) the remaining loan principal is $90\ 000 - 218.96 = 89\ 781.04$ €. Using this value, the corresponding interest payments, principal payments and effective interest payments can be calculated for the second period. This procedure is repeated for the 240 periods until the whole principal of the loan is paid (see Excel solution).

Summation of the present values of the payments of partial collateral, maintenance, loan opening and loan principal and the effective interest rates yields the present value of the investment to be 99 080 € (see the solutions Excel file). This is greater than the present value of the rent 90 915 €. If the increased wealth of the couple is not taken into account, the couple should **not** move in to the new apartment (part a)). Nevertheless, if the increased wealth after paying off the loan is taken into account, the principal payments of the loan are not included in the investment value calculation. This calculation results in the present value of the investment to be 46 747 €, which is significantly less than the present value of the rent. Hence, would the increased wealth be considered, the couple should move in to the new apartment (part b)).

c) The calculations are identical to the earlier part of this exercise apart from the different interest rate $r' = 2.3\%$. Repeating the calculations with new r' yields present values of the rent 115 337 €, the loan w.o. increased wealth considered 109 658 € and the loan with increased wealth considered 38 828 €. Hence at this interest, the couple should move in to the new apartment in both cases.