- The information required by the mean-variance approach is substantial when the number n of assets is large; there are n mean values, n variances, and $n(n-1)/2$ covariances - a total of $2n + n(n-1)/2$ parameters.

- Single-factor model: Suppose that there are n assets, indexed by i, with rates of return $r_i,\ i = 1, 2, \ldots, n$. Suppose then that there is a single factor f, a random quantity, and assume that the rates of return and the factor are related by
$$r_i = a_i + b_i f + e_i, \tag{1}$$
for $i = 1, \ldots, n$. In Equation (1), a_i and b_i are constants and e_i is a random quantity that represents the error of the model. b_i is termed the factor loading that measures the sensitivity of the return to the factor. For the error we assume $\mathbb{E}[e_i] = 0\ \forall i$, because any non-zero mean could be transferred to a_i. Moreover, we assume that the errors are uncorrelated with f and with each other; that is,
$$\mathrm{Cov}[e_i, e_j] = \mathbb{E}[(e_i - \bar{e}_i)(e_j - \bar{e}_j)] = \mathbb{E}[e_i e_j] = 0\ \forall i \neq j$$
$$\mathrm{Cov}[f, e_i] = \mathbb{E}[(f - \bar{f})(e_i - \bar{e}_i)] = \mathbb{E}[(f - \bar{f})e_i] = 0\ \forall i.$$
The variances of the error terms, denoted by $\sigma_{e_i}^2$, are assumed known.

- If we agree to use a single-factor model, the standard parameters for mean-variance analysis can be determined as
$$\bar{r}_i = a_i + b_i \bar{f}$$
$$\sigma_i^2 = b_i^2 \sigma_f^2 + \sigma_{e_i}^2$$
$$\sigma_{ij} = b_i b_j \sigma_f^2,\ i \neq j$$
$$b_i = \frac{\mathrm{Cov}[r_i, f]}{\sigma_f^2}.$$
The only parameters to be estimated for this model are a_i's, b_i's, $\sigma_{e_i}^2$'s, and $\bar{f}$ and σ_f^2's - a total of just $3n + 2$ parameters.

- Suppose that there are n assets with rates of return governed by the single-factor model (1), and suppose that a portfolio of these assets is constructed with weights w_i, with $\sum_{i=1}^n w_i = 1$. Then the rate of return r of the portfolio is
$$r = \sum_{i=1}^n w_i r_i = \sum_{i=1}^n w_i a_i + \sum_{i=1}^n w_i b_i f + \sum_{i=1}^n w_i e_i,$$
and denoting $a = \sum_{i=1}^n w_i a_i$, $b = \sum_{i=1}^n w_i b_i$ and $e = \sum_{i=1}^n w_i e_i$, we can write the above formula as
$$r = a + bf + e.$$
Similarly to the single asset case, a and b are constants and e is a random variable. Because $\mathbb{E}[e_i e_j] = 0, i \neq j$, the variance of e is
$$\sigma_e^2 = \mathbb{E}[e^2] = \mathbb{E}\left[\left(\sum_{i=1}^n w_i e_i\right)\left(\sum_{j=1}^n w_j e_j\right)\right] = \mathbb{E}\left[\sum_{i=1}^n w_i^2 e_i^2\right] = \sum_{i=1}^n w_i^2 \sigma_{e_i}^2, \tag{2}$$

and because $\text{Cov}[f, e_i] = 0 \; \forall i \Rightarrow \text{Cov}[f, e] = 0$, the overall variance of the portfolio is

$$\sigma^2 = \text{Var}[a + bf + e] = b^2\text{Var}[f] + 2b\text{Cov}[f, e] + \text{Var}[e] = b^2\sigma_f^2 + \sigma_e^2. \tag{3}$$

- The CAPM can be interpreted as a special case of a single-factor model. Suppose we model the excess rates of return of stocks $r_i - r_f$ (where r_f is the risk-free rate) with a single-factor model, with the factor being the excess rate of return of the market $r_M - r_f$. Then the factor model becomes

$$r_i - r_f = \alpha_i + \beta_i(r_M - r_f) + e_i.$$

- Arbitrage Pricing Theory (APT) is an alternative theory of asset pricing that can be derived from the factor model framework. Instead of the strong equilibrium assumption of the CAPM theory, APT assumes that
(i) when returns are certain, investors prefer greater return to lesser return,
(ii) the universe of assets being considered is large, and
(iii) there are no arbitrage opportunities.
Specifically, APT assumes that all asset rates of return satisfy a single-factor model with no error term, and hence the uncertainty with a return is due only to the uncertainty in the factor f. For assets i and j, we write

$$r_i = a_i + b_i f \qquad r_j = a_j + b_j f. \tag{4}$$

We then select w such that the coefficient of f in this equation is zero; that is,

$$r = w(a_i + b_i f) + (1 - w)(a_j + b_j f) = wa_i + (1 - w)a_j + \underbrace{(wb_i + (1 - w)b_j)}_{=0} f \Rightarrow w = \frac{b_j}{b_j - b_i}$$

If there is a risk-free asset r_f, the return of this portfolio must have this same rate, and even if there is no explicit risk-free asset, all portfolios constructed this way must have the same rate of return - otherwise there would be an arbitrage opportunity. We denote this rate by λ_0 and find that

$$\begin{aligned} r &= wa_i + (1 - w)a_j = \lambda_0 \\ \Rightarrow& \frac{b_j a_i}{b_j - b_i} - \frac{b_i a_j}{b_j - b_i} = \lambda_0 \\ \Rightarrow& \lambda_0(b_j - b_i) = a_i b_j - a_j b_i \\ \Rightarrow& \frac{a_j - \lambda_0}{b_j} = \frac{a_i - \lambda_0}{b_i} = c. \end{aligned} \tag{5}$$

The last equation holds for all i and j (because we could select any assets i and j that satisfy (**??**)), which is why it must be a constant c. We use this information to write a simple formula for the expected rate of return of asset i as follows:

$$\bar{r}_i = a_i + b_i \bar{f} = \lambda_0 + b_i c + b_i \bar{f} = \lambda_0 + b_i \lambda_1,$$

where $\lambda_1 = c + \bar{f}$ is a constant (same for all assets i). This statement can be generalized for multiple factors:

$$r_i = a_i + \sum_{j=1}^{m} b_{ij} f_j$$
$$\Rightarrow \bar{r}_i = \lambda_0 + \sum_{j=1}^{m} b_{ij} \lambda_j,$$

where the value λ_j is the price of risk associated with the factor f_j, often called the factor price.

- Value = how much a <u>certain</u> event is appreciated
  Utility = how much an <u>uncertain</u> event is appreciated, when making choices associated with risk
  These are concepts of decision theory, which is more extensively studied in course MS-E2134 - Decision making and problem solving.

- A utility function $U(x)$ relates possible wealth levels x to a real value. Once a utility function is defined, all alternative random wealth levels are ranked by evaluating their expected utility values, that is,

$$\mathbb{E}[U(x)] > \mathbb{E}[U(y)] \Leftrightarrow \text{alternative } x \text{ is preferred to alternative } y.$$

- Decision theory assumes non-satiation, which reflects the idea that, every thing else being equal, investors always want more money. Moreover, it is usually assumed that investors are risk averse, that is, investments with the smallest standard deviation for the given mean are preferred. Non-satiation assumption leads to an increasing utility function, and the utility function of a risk averse investor is concave.

- The degree of risk aversion exhibited by a utility function can be formally defined by the Arrow-Pratt absolute risk aversion coefficient, which is

$$a(x) = -\frac{U''(x)}{U'(x)},$$

  where the denominator $U'(x)$ normalizes the coefficient. For many individuals, risk aversion decreases as their wealth increases, and consequently, $a(x)$ is often decreasing.

- The certainty equivalent CE of a random wealth variable x is defined to be the amount of a <u>certain wealth</u> that has a utility level equal to the expected utility of x, that is,

$$U(CE) = \mathbb{E}[U(x)]$$

7.1 (L8.1) (A simple portfolio) Someone who believes that the collection of all stocks satisfies a single-factor model with the market portfolio serving as the factor gives you information on three stocks which make up a portfolio. (See Table 1.) In addition, you know that the market portfolio has an expected rate of return of 12% and a standard deviation of 18%. The risk-free rate is 5%.
a) What is the portfolio's expected rate of return?
b) Assuming the factor model is accurate, what is the standard deviation of this rate of return?

Table 1: Simple Portfolio.

Stock	Beta	Standard deviation of random error term	Weight in portfolio
A	1.10	7.0%	20%
B	0.80	2.3%	50%
C	1.00	1.0%	30%

Solution:

$$r_f = 5\% \qquad r_M = 12\% \qquad \sigma_M = 18\%$$

	β_i	σ_{e_i}	w_i
A	1.10	7.0%	20%
B	0.80	2.3%	50%
C	1.00	1.0%	30%

a) The beta of the portfolio is a weighted combination of the individual betas:

$$\beta_p = w_A\beta_A + w_B\beta_B + w_C\beta_C = 0.92$$

Hence, applying the CAPM to the portfolio we find

$$\bar{r}_p = r_f + \beta_p(\bar{r}_M - r_f) = 0.05 + 0.92(0.12 - 0.05) = 0.1144 = 11.44\%.$$

b) Using formula (**??**) The variance of the error terms of the portfolio is

$$\sigma_e^2 = \sum_{i=1}^{n} w_i^2\sigma_{e_i}^2 = 0.2^2 \times 0.07^2 + 0.5^2 \times 0.023^2 + 0.3^2 \times 0.01^2 = 0.00033725$$

Then, using formula (**??**), the overall variance of the portfolio is

$$\sigma^2 = b^2\sigma_f^2 + \sigma_e^2 = \beta_p^2\sigma_f^2 + \sigma_e^2 = 0.92^2 \times 0.18^2 + 0.00033725 = 0.02776,$$

and the standard deviation of the portfolio is

$$\sigma = \sqrt{\sigma^2} = 0.167 = 16.7\%.$$

7.2 (L8.2) (APT factors) Two stock are believed to satisfy the two-factor model

$$r_1 = a_1 + 2f_1 + f_2$$

$$r_2 = a_2 + 3f_1 + 4f_2.$$

In addition, there is a risk-free asset with a rate of return $r_f = 10\%$. It is known that $\bar{r}_1 = 15\%$ and $\bar{r}_2 = 20\%$. What are the values of λ_0, λ_1 and λ_2 for this model?

Solution:
When the rates of return of the stocks depend on two factors as

$$r_i = a_i + b_{i1}f_1 + b_{i2}f_2,$$

the expected rate of return any asset i is

$$\bar{r}_i = \lambda_0 + b_{i1}\lambda_1 + b_{i2}\lambda_2.$$

Because there is a risk-free asset,

$$\lambda_0 = r_f = 10\%.$$

Because we know the expected rates of return $\bar{r}_i$ of stocks 1 and 2, we can solve λ_1 and λ_2 from the equation system

$$15\% = 10\% + 2\lambda_1 + \lambda_2$$

$$20\% = 10\% + 3\lambda_1 + 4\lambda_2.$$

Solving this yields

$$\lambda_1 = 2\% \qquad \lambda_2 = 1\%.$$

Hence the prices of risks of factors 1 and 2 are 2% and 1%, respectively.

7.3 (L8.4) (Variance estimate) Let r_i, for $i = 1, 2, \ldots, n$, be independent samples of a return r of mean $\bar{r}$ and variance σ^2. Define the estimates

$$\hat{\bar{r}} = \frac{1}{n}\sum_{i=1}^{n} r_i$$

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(r_i - \hat{\bar{r}})^2.$$

Show that $\mathbb{E}[s^2] = \sigma^2$.

Solution:
We show that s^2 is an unbiased estimate of the variance. First we write the formula of the average $\hat{\bar{r}}$ in the formula of the sample variance. Then we add and subtract the true expected rate of return $\bar{r}$ inside the squared term. These modifications yield

$$\mathbb{E}[s^2] = \mathbb{E}\left[\frac{1}{n-1}\sum_{i=1}^{n}(r_i - \hat{\bar{r}})^2\right] = \mathbb{E}\left[\frac{1}{n-1}\sum_{i=1}^{n}\left(r_i - \frac{1}{n}\sum_{j=1}^{n} r_j\right)^2\right] = \mathbb{E}\left[\frac{1}{n-1}\sum_{i=1}^{n}\left((r_i - \bar{r}) - \frac{1}{n}\sum_{j=1}^{n}(r_j - \bar{r})\right)^2\right]$$

Extracting r_i from the inner summation and moving the factor $1/(n-1)$ outside the expected value yields

$$\frac{1}{n-1}\mathbb{E}\left[\sum_{i=1}^{n}\left(\left(1 - \frac{1}{n}\right)(r_i - \bar{r}) - \frac{1}{n}\sum_{j\neq i}(r_j - \bar{r})\right)^2\right].$$

We then expand the square inside the summation to get

$$\frac{1}{n-1}\mathbb{E}\left[\sum_{i=1}^{n}\left(\left(1 - \frac{1}{n}\right)^2(r_i - \bar{r})^2 - 2\left(1 - \frac{1}{n}\right)(r_i - \bar{r})\frac{1}{n}\sum_{j\neq i}(r_j - \bar{r}) + \left(\frac{1}{n}\sum_{j\neq i}(r_j - \bar{r})\right)^2\right)\right].$$

Moving the expected value inside the summation and expanding the term $\left[(1/n)\sum_{j\neq i}(r_j - \bar{r})\right]^2$ yields

$$\frac{1}{n-1}\sum_{i=1}^{n}\left[\left(1 - \frac{1}{n}\right)^2\mathbb{E}\left[(r_i - \bar{r})^2\right] - 2\left(1 - \frac{1}{n}\right)\frac{1}{n}\sum_{j\neq i}\mathbb{E}\left[(r_i - \bar{r})(r_j - \bar{r})\right] + \frac{1}{n^2}\sum_{j\neq i}\sum_{k\neq i}\mathbb{E}\left[(r_j - \bar{r})(r_k - \bar{r})\right]\right].$$

The samples were assumed independent and hence have zero covariance, that is, $\mathbb{E}\left[(r_i - \bar{r})(r_j - \bar{r})\right] = \sigma_{ij} = 0,\ i \neq j$. Note also that $\mathbb{E}\left[(r_i - \bar{r})^2\right] = \sigma^2$. These modifications give the above formula as

$$\frac{1}{n-1}\sum_{i=1}^{n}\left[\left(1 - \frac{1}{n}\right)^2\sigma^2 + \frac{1}{n^2}(n-1)\sigma^2\right] = \sigma^2\frac{n}{n-1}\left[\left(1 - \frac{1}{n}\right)^2 + \frac{1}{n^2}(n-1)\right].$$

This can be simplified into

$$\sigma^2\frac{n}{n-1}\left[1 - \frac{2}{n} + \frac{1}{n^2} + \frac{1}{n} - \frac{1}{n^2}\right] = \sigma^2\frac{n}{n-1}\left(1 - \frac{1}{n}\right) = \sigma^2\frac{n-1}{n-1} = \sigma^2$$

Thus, we have shown that $\mathbb{E}[s^2] = \sigma^2$.
Note that if the true expected value $\bar{r}$ was known, the denominator of the variance estimate would be n instead of $n-1$, because $\mathbb{E}\left[(1/n)\sum_{i=1}^{n}(r_i - \bar{r})^2\right] = (1/n)\sum_{i=1}^{n}\mathbb{E}\left[(r_i - \bar{r})^2\right] = (1/n)n\sigma^2 = \sigma^2$.

7.4 (L8.7) (Clever, but no cigar) Kalle Virtanen figured out a clever way to get 24 samples of monthly returns in just over one year instead of only 12 samples; he takes overlapping samples; that is, the first sample covers Jan 1 to Feb 1, and the second sample covers Jan 15 to Feb 15, and so forth. He figures that the error in his estimate of $\bar{r}$, the mean monthly return, will be reduced by this method. Analyse Kalle's idea. How does the variance of his estimate compare with that of the usual method of using 12 non-overlapping monthly returns?

Solution:

First divide the year into half-month intervals and index these time points by i. Let r_i be the return over the i-th full month (but some will start midway through the month). We let $\bar{r}_m$ and σ_m^2 denote the monthly expected return and variance of that return.

Now let ρ_i be the return over the i-th half-month period. Assume that these returns are uncorrelated. The return over any monthly period is a sum of two half-month returns; that is, the monthly return r_i is $r_i = \rho_i + \rho_{i+1}$. It is easy to see that

$$\bar{r}_i = \bar{\rho}_i + \bar{\rho}_{i+1} \Rightarrow \bar{r}_m = 2\bar{\rho} \Rightarrow \bar{\rho} = \bar{r}_m/2$$

$$\sigma_m^2 = \mathrm{Var}[\rho_i + \rho_{i+1}] = \mathrm{Var}[\rho_i] + \mathrm{Var}[\rho_{i+1}] = 2\sigma_\rho^2 \Rightarrow \sigma_\rho^2 = \sigma^2/2,$$

where $\bar{\rho}$ is the expected half-monthly return and σ_ρ^2 is the variance of that return. The covariance of two monthly returns r_i and r_j is then

$$\begin{aligned}
\mathrm{Cov}[r_i, r_j] &= \mathrm{Cov}[\rho_i + \rho_{i+1}, \rho_j + \rho_{j+1}] \\
&= \mathrm{Cov}[\rho_i, \rho_j] + \mathrm{Cov}[\rho_i, \rho_{j+1}] + \mathrm{Cov}[\rho_{i+1}, \rho_j] + \mathrm{Cov}[\rho_{i+1}, \rho_{j+1}] \\
\Rightarrow \mathrm{Cov}[r_i, r_j] &= \begin{cases} \mathrm{Var}[r_i] = \sigma^2, & \text{if } i = j \\ \mathrm{Var}[\rho_i] = \sigma^2/2, & \text{if } |i-j| = 1 \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Now for Kalle's scheme we form the estimate

$$\hat{\bar{r}} = \frac{1}{24}\sum_{i=1}^{24} r_i.$$

We need to evaluate

$$\begin{aligned}
\mathrm{Var}\left[\hat{\bar{r}}\right] &= \mathrm{Var}\left[\frac{1}{24}\sum_{i=1}^{24} r_i\right] = \frac{1}{24^2}\sum_{i=1}^{24}\sum_{j=1}^{24}\mathrm{Cov}\left[r_i, r_j\right] \\
&= \frac{1}{24^2}\left[\mathrm{Var}[r_1] + \mathrm{Cov}[r_1, r_2] + \sum_{j=2}^{23}\left(\mathrm{Cov}[r_{j-1}, r_j] + \mathrm{Var}[r_j] + \mathrm{Cov}[r_j, r_{j+1}]\right) + \mathrm{Cov}[r_{23}, r_{24}] + \mathrm{Var}[r_{24}]\right]
\end{aligned}$$

Hence we have

$$\mathrm{Var}\left[\hat{\bar{r}}\right] = \frac{1}{24^2}\left[24\sigma^2 + 2\times 23\sigma^2/2\right] = \frac{1}{24^2}\left[24\sigma^2 + 24\sigma^2 - \sigma^2\right] = \frac{2\times 24}{24^2}\sigma^2 - \frac{1}{24^2}\sigma^2 = \frac{1}{12}\sigma^2 - \frac{1}{24^2}\sigma^2.$$

For twelve non-overlapping months of data we would have

$$\mathrm{Var}[\hat{\bar{r}}] = \mathrm{Var}\left[\frac{1}{12}\sum_{i=1}^{12} r_i\right] = \frac{12}{12^2}\sigma^2 = \frac{1}{12}\sigma^2.$$

The difference $\sigma^2/24^2 \approx 0.0017\sigma^2$ is caused by the longer estimation period for the half-month case; the last sample is actually partly from the following year, because sample r_{24} covers Dec 15 to Jan 15. Hence the estimation precision cannot be increased by sampling with half-month intervals; the gain of greater sample size is lost because of the correlation of the overlapping samples.

7.5 (L9.1) (Certainty equivalent) An investor has utility function $U(x) = x^{1/4}$ for salary. He has a new job offer which pays 80000 € with a bonus. The bonus will be 0 €, 10000 €, 20000 €, 30000 €, 40000 €, 50000 €, or 60000 €, each with equal probability. What is the certainty equivalent of this job offer?

Solution:
The certainty equivalent can be solved by taking the inverse utility function from the expected utility of a random wealth variable, because

$$U\left(CE[X]\right) = \mathbb{E}\left[U(X)\right] \Rightarrow CE[X] = U^{-1}\left(\mathbb{E}[U(X)]\right)$$

$$U(x) = x^{1/4} \Rightarrow U^{-1}(x) = x^4$$

The possible incomes and their utility levels (found by taking the 1/4-th power) are

Bonus	Total income	$U(x)$	$p(x)$	$p(x)U(x)$
0	80000	16.82	1/7	2.40
10000	90000	17.32	1/7	2.47
20000	100000	17.78	1/7	2.54
30000	110000	18.21	1/7	2.60
40000	120000	18.61	1/7	2.66
50000	130000	18.99	1/7	2.71
60000	140000	19.34	1/7	2.76
$\sum$				18.15

Hence

$$\mathbb{E}\left[U(X)\right] = 18.15 \Rightarrow CE[X] = U^{-1}\left(\mathbb{E}[U(X)]\right) = 18.15^4 = 108610e.$$