- A forward contract on a commodity is a contract to purchase or sell a specific amount of the commodity at a specific price and at a specific time in the future. The cash flows of a forward contract occur at the time of delivery, that is, at the end of the contract.

- The buyer is said to be long, and the seller is said to be short. Being long or short a given amount is the position of the party. In general, the party which benefits from the value increase of an asset is said to be long, and the party which benefits from the value decrease of the asset is short.

- The forward price is the delivery price of the commodity to be delivered at a specific future date. This price is almost always negotiated so that the initial payment is zero; that is, the value of the contract is zero when it is initiated.
  The forward prices can be determined using the forward rates (the rates of interest associated with an agreement to loan money over a specified interval of time in the future). Suppose first that a commodity can be stored at zero cost and also sold short. Then suppose that we

  1. buy one unit of the commodity at price S in the spot market,
  2. enter a forward contract to deliver at time T one unit at price F (that is, we short one unit), and
  3. store the commodity until T and then deliver it to meet our obligation and obtain F.

  The present value of the stream $(-S, F)$ must be zero, so that the value of the contract is zero when initiated. Suppose that the discount factor between 0 and T is $d(0,T)$. Then the forward price is

  $$-S + d(0,T)F = 0 \Rightarrow F = \frac{S}{d(0,T)}.$$

  Suppose now that the holding cost of holding an asset in the period from k to $k+1$ is $c(k)$ per unit (paid in the beginning of the period). Suppose then that we buy one unit at price S, enter a forward contract at time T and price F, and then store the commodity until T and obtain F at time T. We denote by M the number of periodic holding payments to be paid until the time of delivery. The associated cash flow stream is $(-S - c(0), -c(1), -c(2), \ldots, -c(M-1), F)$. The present value of this stream must be zero and hence

  $$-S + -\sum_{k=0}^{M-1} d(0,k)c(k) + d(0,M)F = 0 \Rightarrow F = \frac{S}{d(0,M)} + \sum_{k=0}^{M-1} \frac{c(k)}{d(k,M)}, \tag{1}$$

  where $d(k,M)$ is the discount factor from k to M. Note that $c(k)$ can be negative (e.g., if the asset is a bond that pays periodic coupon payments).

- The value of a forward contract changes in time. Suppose that a forward contract made in the past for delivery at time T in the future has a delivery price F_0, and that the current (at present time t) forward price for delivery at time T is F_t. The value of the contract is

  $$f_t = (F_t - F_0)d(t,T), \tag{2}$$

  where $d(t,T)$ is the risk-free discount factor over the period from t to T.

8.1 (L10.1) (Gold futures) The current price of gold is 412 € per ounce. The storage cost is 2 € per ounce per year, payable quarterly in advance. Assuming a constant interest rate of 9% compounded quarterly, what is the theoretical forward price of gold for delivery in 9 months?

Solution:
Applying Equation (**??**) we have

$$\begin{aligned} F &= \frac{S}{d(0,M)} + \sum_{k=0}^{M-1} \frac{c(k)}{d(k,M)} \\ &= \frac{412}{d(0,3)} + \frac{2/4}{d(0,3)} + \frac{2/4}{d(1,3)} + \frac{2/4}{d(2,3),} \end{aligned}$$

where k is the number of 3-month periods. Because constant interest rate implies constant forward rates, $r_{3m} = 9\%/4$ for any 3-month period in the future. Hence

$$\begin{aligned} d(0,3) &= \frac{1}{(1+0.09/4)^3} = 0.9354 \\ d(1,3) &= \frac{1}{(1+0.09/4)^2} = 0.9565 \\ d(2,3) &= \frac{1}{1+0.09/4} = 0.9780. \end{aligned}$$

Combining these, we find

$$F = 440.45 + 0.5345 + 0.5227 + 0.5112 = 442.02e.$$

8.2 (L10.2) (Proportional carrying charges) Suppose that a forward contract of an asset is written at time zero and there are M periods until delivery. Suppose that the carrying charge in period k is $qS(k)$ ($q \in (0,1)$) where $S(k)$ is the spot price of the asset in period k. Show that the forward price is

$$F = \frac{S}{(1-q)^M d(0,M)}.$$

Solution:
To prove the above formula, we make the following arrangements.

Suppose that we are at the sellers side of the contract, that is, we have the short position in the forward contract. We make such a contract that we are committed to sell $(1-q)^M$ commodity at the end of M periods for price F.

At time zero we take a loan for $S(0)$ to purchase 1 unit of the commodity. Then, to pay the carrying charges $qS(k)$ in each period k, in each period we sell a fraction q of the current amount of our commodity holdings. Hence, we make the following procedure.

- In period $k = 0$, we have 1 commodity remaining. We sell a fraction q of these commodities to obtain cash to pay the carrying charge $qS(0)$.
- In period $k = 1$, we have $(1-q)$ commodity remaining. We sell a fraction q of these commodities to obtain cash to pay the carrying charge $q(1-q)S(1)$.
- In period $k = 2$, we have $(1-q)(1-q) = (1-q)^2$ commodity remaining. We sell a fraction q of these commodities to obtain cash to pay the carrying charge $q(1-q)^2S(2)$.

$\vdots$

- In period $k = M-1$, we have $\prod_{j=0}^{M-1}(1-q) = (1-q)^{M-1}$ commodity remaining. We sell a fraction q of these commodities to obtain cash to pay the carrying charge $q(1-q)^{M-1}S(M-1)$.
- In period $k = M$, we have $\prod_{j=0}^{M}(1-q) = (1-q)^M$ commodity remaining.

We then deliver our commodity holdings to receive $F(1-q)^M$, and repay our loan by paying $S(0)/d(0,M)$. The total profit of these transactions (happening at the end of M periods) is

$$F(1-q)^M - S(0)/d(0,M),$$

and to avoid arbitrage, this profit must be zero. Hence

$$F = \frac{S(0)}{(1-q)^M d(0,M)}.$$

8.3 (L10.7) (A bond forward) A certain 10-year bond is currently selling for 920 €. A friend of yours owns a forward contract on this bond that has a delivery date in 1 year and a delivery price of 940 €. The bond pays coupons of 80 € every 6 months, with one due 6 month from now and another just before maturity of the forward. The current interest rates for 6 months and 1 year (compounded semiannually) are 7% and 8%, respectively (annual rates compounded every 6 months). What is the current value of the forward contract?

Solution:
The value of a forward contract is $f_t = (F_t - F_0)d(t, T)$ according to Equation (**??**).
Using the spot rates s_k, we first calculate the discount factors from time 0 to times $k = 1, 2$, where k is the number of 6-month periods.

$$d(0,1) = \frac{1}{1 + s_1/2} = \frac{1}{1.035}$$
$$d(0,2) = \frac{1}{(1 + s_2/2)^2} = \frac{1}{1.04^2}$$

We then solve for F_t the current forward price of the bond using Equation (**??**):

$$F_t = \frac{S}{d(0,M)} + \sum_{k=0}^{M-1} \frac{c(k)}{d(k,M)} = \frac{S}{d(0,2)} + \sum_{k=0}^{1} \frac{d(0,k)c(k)}{d(0,2)}$$
$$= 920(1.04)^2 - \frac{80(1.04)^2}{1.035} - \frac{80(1.04^2)}{1.04^2} = 831.47e,$$

where the second equality follows from $d(0, M) = d(0, k)d(k, M)$, for any k.
Now we solve the value of the forward contract:

$$f_t = (F_t - F_0)d(0,2) = \frac{831.47 - 940}{1.04^2} = -100.34e.$$

Holding the contract causes an obligation to buy the bond at price $F_0 > F_t$ at the time of delivery. Because a smaller price F_t could be negotiated today for the same delivery date, it is unprofitable to hold this contract.