- A forward contract on a commodity is a contract to purchase or sell a specific amount of the commodity at a specific price and at a specific time in the future. The cash flows of a forward contract occur at the time of delivery, that is, at the end of the contract.
- Futures market is an organized exchange on forward contracts. Individual futures contracts are made with the exchange, not between individual traders. The delivery dates and quantities of the future markets can be standardized in a straightforward manner. However, standardization of forward prices is impossible, because the forward contracts with a specified delivery date T have a price F_t that changes continuously. Instead of standardizing the forward prices, futures market revises each individual contract as the price environment changes. The change in the price of each contract is compensated in a margin account; if a new price is more favourable for the trader, a payment is made from the margin account, and if the new price is inferior, a compensation is paid in the account.
- If interest rates follow expectations dynamics, the theoretical futures and forward prices of corresponding contracts are identical.
- The primary use of futures contracts is to hedge against risk.
- The simplest hedging strategy is the perfect hedge. This is made by taking a futures market position that has an equal and opposite price risk than the hedged future commitment.
  Suppose that your company has received an order in 90 days that will be paid in dollars; that is, your company is receiving dollars and is currently long in the dollar market. Your company can then hedge the contract from foreign exchange risk by going short in the dollar market. (Viewed alternatively, after receiving the dollars, your company wants to sell them, so to sell the dollars early, a short position should be taken.)
- It is not always possible to form a perfect hedge with futures contracts. For example, there may be no contract involving the exact asset whose value must be hedged.
- Minimum variance hedge is made by taking a position in futures market that minimizes the variance of the cash flow at the time of the hedged commitment.
  Suppose that W commodities are sold at time T at the spot price S_T, and hence the cash flow from the commitment is $x = WS_T$. Let F denote the futures price of the contract used as a hedge and h the futures position taken. The total payments in the margin account (neglecting interest payments) until time T are $(F_T - F_0)h$. Hence the total cash flow at the time of delivery is

  $$y = x + (F_T - F_0)h.$$

  The variance of the final cash flow $\mathrm{Var}[y]$ is minimized with the minimum variance hedge

  $$h = -\frac{\mathrm{Cov}[x, F_T]}{\mathrm{Var}[F_T]} = -\frac{\mathrm{Cov}[S_T, F_T]}{\mathrm{Var}[F_T]}W.$$

  Note that the covariance and variance in this formula are expressed in terms of monetary values (unlike the common convention of expressing the volatilities of stocks as a relative value).

- A swap is an agreement to exchange one cash flow stream for another. The most common is the plain vanilla swap, in which one party swaps a series of fixed-level payments for a series of variable level payments. Swaps can be seen as a series of forward contracts in that the fixed level payments corresponds the forward prices and the variable level payments corresponds the spot prices of a commodity.

9.1 (L10.13) (Barley hedge) Farmer Ville Virtanen has a crop of barley that will be ready for harvest and sale in 3 months. The size of the crop is 150kg. Virtanen is worried about possible price changes, so he is considering hedging. There is no futures contract available for barley, but there is futures contract for wheat. His son, Kalle, recently studied minimum-variance hedging and suggests it as a possible approach. Currently the spot prices are 0.20 € per kilo for wheat and 0.30 € for barley. The standard deviations of the prices of both wheat and barley are about 20% per year, and the correlation coefficient between them is about 0.7. What is the minimum-variance hedge for farmer Virtanen, and how effective is this hedge as compared to no hedge?

Solution:

We find such a position h in wheat futures that minimizes the cash flow the farmer receives after 3 months. The cash flow y of the barley proceeds and payments from futures contract at time $T = 3m$ is

$$y = S_T^b W + (F_T - F_0)h,$$

where S_T^b is the spot price of barley and F_T the forward price of wheat at time T, and W is the size of the crop and F_0 the forward price at time 0. Assuming that there is a wheat futures contract with delivery in $T = 3m$, the forward price F_T is the spot price of wheat at time T, S_T^w. Hence the variance of the cash flow y is

$$\mathrm{Var}[y] = \mathrm{Var}[S_T^b W + (F_T - F_0)h] = W^2 \mathrm{Var}[S_T^b] + h^2 \mathrm{Var}[S_T^w] + 2hW \mathrm{Cov}[S_T^b, S_T^w]. \tag{1}$$

We minimize this by solving the optimal h^* that sets the derivative of variance to zero. Hence

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}h}\mathrm{Var}[y]\big|_{h=h^*} &= 2h^* \mathrm{Var}[S_T^w] + 2W \mathrm{Cov}[S_T^b, S_T^w] = 0 \\ \Rightarrow h^* &= -\frac{W \mathrm{Cov}[S_T^b, S_T^w]}{\mathrm{Var}[S_T^w]} = -\frac{W\sigma_{bw}}{\sigma_w^2} = -\frac{W\rho\sigma_b\sigma_w}{\sigma_w^2} \\ \Rightarrow h^* &= -\rho\frac{\sigma_b}{\sigma_w}W. \end{aligned} \tag{2}$$

The relative volatility of both wheat and barley are 20% and the initial spot prices are $S_0^b = 0.3\,e, S_0^w = 0.2\,e$. We calculate the standard deviations of the spot prices as $\sigma_b = 0.2 \cdot S_0^b = 0.06\,e$ and $\sigma_w = 0.2 \cdot S_0^w = 0.04\,e$. Substituting these and $W = 150$ kg, $\rho = 0.7$ into (2) gives the minimum variance hedge

$$h^* = -0.7 \cdot \frac{0.06\,e}{0.04\,e} 150 \text{ kg} = -157.50 \text{ kg}.$$

We substitute the optimal position (2) to (1) and find the minimum variance of cash flow y as

$$\begin{aligned} \mathrm{Var}[y]\big|_{h=h^*} &= W^2\sigma_b^2 + \left(-\rho\frac{\sigma_b}{\sigma_w}W\right)^2 \sigma_w^2 + 2\left(-\rho\frac{\sigma_b}{\sigma_w}W\right) W\rho\sigma_b\sigma_w \\ &= \left(1 + \rho^2 - 2\rho^2\right) W^2\sigma_b^2 = \left(1 - \rho^2\right) W^2\sigma_b^2. \end{aligned}$$

Hence the ratio of variances and standard deviations are

$$\frac{\mathrm{Var}[y]\big|_{h=h^*}}{\mathrm{Var}[WS_T^b]} = \frac{\left(1-\rho^2\right)W^2\sigma_b^2}{W^2\sigma_b^2} = 0.51 \text{ and } \frac{\mathrm{Std}[y]\big|_{h=h^*}}{\mathrm{Std}[WS_T^b]} = \frac{\sqrt{1-\rho^2}W\sigma_b}{W\sigma_b} = \sqrt{1-\rho^2} = 0.71.$$

Thus, the standard deviation of the cash flow is roughly 30% smaller with the minimum variance hedge. Note that the expected value of the hedged cash flow may be smaller than that of the unhedged cash flow.

9.2 (L10.14) (Opposite hedge variance) Assume that cash flow is given by $y = S_T W + (F_T - F_0)h$. Let $\sigma_S^2 = \mathrm{Var}[S_T], \sigma_F^2 = \mathrm{Var}[F_T]$, and $\sigma_{ST} = \mathrm{Cov}[S_T, F_T]$.
In an equal and opposite hedge, h is taken to be an opposite equivalent euro value of the hedging instrument. Therefore $h = -kW$, where k is the price ratio between the asset and the hedging instrument. Express the standard deviation of y with the equal and opposite hedge in the form

$$\sigma_y = W\sigma_S \times B.$$

(That is, find B.) What does B represent?

Solution:

Substituting $h = -kW$ yields the cash flow at time T as

$$y = S_T W + (F_T - F_0)h = S_T W - kW(F_T - F_0).$$

The variance of y can be written as

$$\begin{aligned}
&\mathrm{Var}[y] = W^2\mathrm{Var}[S_T] + k^2W^2\mathrm{Var}[F_T] - 2kW^2\mathrm{Cov}[S_T, F_T] \\
&\Rightarrow \sigma_y^2 = W^2\left(\sigma_S^2 + k^2\sigma_F^2 - 2k\sigma_{ST}\right) \\
&\qquad = W^2\sigma_S^2\left(1 + k^2\frac{\sigma_F^2}{\sigma_S^2} - 2k\frac{\sigma_{ST}}{\sigma_S^2}\right) \\
&\Rightarrow \sigma_y = W\sigma_S\sqrt{1 + k^2\frac{\sigma_F^2}{\sigma_S^2} - 2k\frac{\sigma_{ST}}{\sigma_S^2}}
\end{aligned}$$

Hence

$$B = \sqrt{1 + k^2\frac{\sigma_F^2}{\sigma_S^2} - 2k\frac{\sigma_{ST}}{\sigma_S^2}}.$$

$B = \sigma_y/(W\sigma_S)$ is the ratio of the volatility of the cash flow under hedge and without hedge. That is, it represents the effectiveness of the hedge.

For minimum variance hedge the ratio $B = \sigma_y/(W\sigma_S)$ is (see previous exercise)

$$B = \sqrt{1 - \rho_{ST}^2} = \sqrt{1 - \left(\frac{\sigma_{ST}}{\sigma_F\sigma_S}\right)^2}$$

If $B = 0$, the price risk is eliminated entirely (which occurs for example if $F_T = S_T$), and such a hedging instrument is termed "perfect match".

9.3 (L10.11) (Specific vanilla) Suppose the current term structure of interest rates is (0.070, 0.073, 0.077, 0.081, 0.084, 0.088). A plain vanilla interest rate swap will make payments at the end of each year equal to the floating short rate that was posted at the beginning of that year. A 6-year swap having a notional principal of 10 million € is being configured.
a) What is the value of the floating rate portion of the swap?
b) What rate of interest for the fixed portion of the swap would make the two sides of the swap equal?

Solution:

a) The floating rate portion of the swap receives the interest payments according to the floating short rates. Consider a swap that lasts M periods where in each period party A gets proceeds according to the floating rates determined in the previous period. Assuming expectations dynamics, these are on expectation the short rates r_{i-1}. The interest charges are made according to a notional principal of $P = 10Me$, and hence the floating payments are $r_{i-1}P, i = 1, 2, \ldots, M$. The present value of the floating side of the swap is worth

$$\begin{aligned} V_{float} &= \sum_{i=1}^{M} d(0,i) r_{i-1} P \\ &= \sum_{i=1}^{M} \frac{1}{(1+r_0)(1+r_1)\cdots(1+r_{i-1})} r_{i-1} P \\ &= \sum_{i=1}^{M} \left[\frac{1+r_{i-1}}{(1+r_0)(1+r_1)\cdots(1+r_{i-1})} - d(0,i) \right] P \\ &= \sum_{i=1}^{M} \left[d(0,i-1) - d(0,i) \right] P \\ &= \left[1 - d(0,M) \right] P. \end{aligned}$$

Because $M = 6$, and $d(0,6) = 1/1.088^6 = 0.6029$ as implied by the term structure, we have

$$V_{float} = [1 - 0.6029] \cdot 10\text{M}e = 3.971\text{M}e.$$

b) The fixed rate portion of the swap receives interest payments according to a fixed rate r. Hence

$$\begin{aligned} V_{fixed} &= \sum_{i=1}^{6} d(0,i) r P \\ &= 4.607 r \cdot 10\text{M}e, \end{aligned}$$

where 4.607 was obtained by summing the discount rates implied by the term structure. Setting $V_{fixed} = V_{float}$, we find

$$r = 0.0862 = 8.62\%.$$

Party A makes payments according to the fixed rate r in exchange for the floating rate stream. By making the values of both sides of the contract equal, the swap will be beneficial for both parties and arbitrage opportunities are eliminated.