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Chapter 9

Auctions

In Chapter 8, we considered a first extended application of game-theoretic ideas, in our analysis of traffic flow through a network. Here we consider a second major application — the behavior of buyers and sellers in an auction.

An auction is a kind of economic activity that has been brought into many people's everyday lives by the Internet, through sites such as eBay. But auctions also have a long history that spans many different domains. For example, the U.S. government uses auctions to sell Treasury bills and timber and oil leases; Christie's and Sotheby's use them to sell art; and Morrell & Co. and the Chicago Wine Company use them to sell wine.

Auctions will also play an important and recurring role in the book, since the simplified form of buyer-seller interaction they embody is closely related to more complex forms of economic interaction as well. In particular, when we think in the next part of the book about markets in which multiple buyers and sellers are connected by an underlying network structure, we'll use ideas initially developed in this chapter for understanding simpler auction formats. Similarly, in Chapter 15, we'll study a more complex kind of auction in the context of a Web search application, analyzing the ways in which search companies like Google, Yahoo!, and Microsoft use an auction format to sell advertising rights for keywords.

9.1 Types of Auctions

In this chapter we focus on different simple types of auctions, and how they promote different kinds of behavior among bidders. We'll consider the case of a seller auctioning one item to a set of buyers. We could symmetrically think of a situation in which a buyer is trying to purchase a single item, and runs an auction among a set of multiple sellers, each of whom is able to provide the item. Such *procurement auctions* are frequently run by governments to

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purchase goods. But here we'll focus on the case in which the seller runs the auction.

There are many different ways of defining auctions that are much more complex than what we consider here. The subsequent chapters will generalize our analysis to the case in which there are multiple goods being sold, and the buyers assign different values to these goods. Other variations, which fall outside the scope of the book, include auctions in which goods are sold sequentially over time. These more complex variations can also be analyzed using extensions of the ideas we'll talk about here, and there is a large literature in economics that considers auctions at this broad level of generality [256, 292].

The underlying assumption we make when modeling auctions is that each bidder has an *intrinsic value* for the item being auctioned; she is willing to purchase the item for a price up to this value, but not for any higher price. We will also refer to this intrinsic value as the bidder's *true value* for the item. There are four main types of auctions when a single item is being sold (and many variants of these types).

1. *Ascending-bid auctions*, also called *English auctions*. These auctions are carried out interactively in real time, with bidders present either physically or electronically. The seller gradually raises the price, bidders drop out until finally only one bidder remains, and that bidder wins the object at this final price. Oral auctions in which bidders shout out prices, or submit them electronically, are forms of ascending-bid auctions.

2. *Descending-bid auctions*, also called *Dutch auctions*. This is also an interactive auction format, in which the seller gradually lowers the price from some high initial value until the first moment when some bidder accepts and pays the current price. These auctions are called Dutch auctions because flowers have long been sold in the Netherlands using this procedure.

3. *First-price sealed-bid auctions*. In this kind of auction, bidders submit simultaneous "sealed bids" to the seller. The terminology comes from the original format for such auctions, in which bids were written down and provided in sealed envelopes to the seller, who would then open them all together. The highest bidder wins the object and pays the value of her bid.

4. *Second-price sealed-bid auctions*, also called *Vickrey auctions*. Bidders submit simultaneous sealed bids to the sellers; the highest bidder wins the object and pays the value of the second-highest bid. These auctions are called Vickrey auctions in honor of William Vickrey, who wrote the first game-theoretic analysis of auctions (including the second-price auction [400]). Vickery won the Nobel Memorial Prize in Economics in 1996 for this body of work.

9.2 When are Auctions Appropriate?

Auctions are generally used by sellers in situations where they do not have a good estimate of the buyers' true values for an item, and where buyers do not know each other's values. In this case, as we will see, some of the main auction formats can be used to elicit bids from buyers that reveal these values.

Known Values. To motivate the setting in which buyers' true values are unknown, let's start by considering the case in which the seller and buyers know each other's values for an item, and argue that an auction is unnecessary in this scenario. In particular, suppose that a seller is trying to sell an item that he values at x, and suppose that the maximum value held by a potential buyer of the item is some larger number y. In this case, we say there is a *surplus* of $y - x$ that can be generated by the sale of the item: it can go from someone who values it less (x) to someone who values it more (y).

If the seller knows the true values that the potential buyers assign to the item, then he can simply announce that the item is for sale at a fixed price just below y, and that he will not accept any lower price. In this case, the buyer with value y will buy the item, and the full value of the surplus will go to the seller. In other words, the seller has no need for an auction in this case: he gets as much as he could reasonably expect just by announcing the right price.

Notice that there is an asymmetry in the formulation of this example: we gave the seller the ability to commit to the mechanism that was used for selling the object. This ability of the seller to "tie his hands" by committing to a fixed price is in fact very valuable to him: assuming the buyers believe this commitment, the item is sold for a price just below y, and the seller makes all the surplus. In contrast, consider what would happen if we gave the buyer with maximum value y the ability to commit to the mechanism. In this case, this buyer could announce that she is willing to purchase the item for a price just above the larger of x and the values held by all other buyers. With this announcement, the seller would still be willing to sell — since the price would be above x — but now at least some of the surplus would go to the buyer. As with the seller's commitment, this commitment by the buyer also requires knowledge of everyone else's values.

These examples show how commitment to a mechanism can shift the power in the transaction in favor of the seller or the buyer. One can also imagine more complex scenarios in which the seller and buyers know each other's values, but neither has the power to unilaterally commit to a mechanism. In this case, one may see some kind of bargaining take place over the price; we discuss the topic of bargaining further in Chapter 12. As we will discover in the current chapter, the issue of commitment is also crucial in the context of auctions — specifically, it is important that a seller be able to reliably commit in advance to a given auction format.

Unknown Values. Thus far we've been discussing how sellers and buyers might interact when everyone knows each other's true values for the item. Beginning in the next section, we'll see how auctions come into play when the participants do not know each other's values.

For most of this chapter we will restrict our attention to the case in which the buyers have *independent, private values* for the item. That is, each buyer knows how much she values the item, she does not know how much others value it, and her value for it does not depend on others' values. For example, the buyers could be interested in consuming the item, with their values reflecting how much they each would enjoy it.

Later we will also consider the polar opposite of this setting — the case of *common values*. Suppose that an item is being auctioned, and instead of consuming the item, each buyer plans to resell the item if she gets it. In this case (assuming the buyers will do a comparably good job of reselling it), the item has an unknown but common value regardless of who acquires it: it is equal to how much revenue this future reselling of the item will generate. Buyers' estimates of this revenue may differ if they have some private information about the common value, and so their valuations of the item may differ. In this setting, the value each buyer assigns to the object would be affected by knowledge of the other buyers' valuations, since the buyers could use this knowledge to further refine their estimates of the common value.

9.3 Relationships between Different Auction Formats

Our main goal will be to consider how bidders behave in different types of auctions. We begin in this section with some simple, informal observations that relate behavior in interactive auctions (ascending-bid and descending-bid auctions, which play out in real time) with behavior in sealed-bid auctions. These observations can be made mathematically rigorous, but for the discussion here we will stick to an informal description.

Descending-Bid and First-Price Auctions. First, consider a descending-bid auction. Here, as the seller is lowering the price from its high initial starting point, no bidder says anything until finally someone actually accepts the bid and pays the current price. Bidders therefore learn nothing while the auction is running, other than the fact that no one has yet accepted the current price. For each bidder i, there's a first price b_i at which she'll be willing to break the silence and accept the item at price b_i. So with this view, the process is equivalent to a sealed-bid first-price auction: this price b_i plays the role of bidder i's bid; the item goes to the bidder with the highest bid value; and this bidder pays the value of her bid in exchange for the item.

Ascending-Bid and Second-Price Auctions. Now let's think about an ascending-bid auction, in which bidders gradually drop out as the seller steadily raises the price. The winner of the auction is the last bidder remaining, and she pays the price at which the second-to-last bidder drops out.[1]

Suppose that you're a bidder in such an auction; let's consider how long you should stay in the auction before dropping out. First, does it ever make sense to stay in the auction after the price reaches your true value? No: by staying in, you either lose and get nothing, or else you win and have to pay more than your value for the item. Second, does it ever make sense to drop out before the price reaches your true value for the item? Again, no: if you drop out early (before your true value is reached), then you get nothing, when by staying in you might win the item at a price below your true value.

So this informal argument indicates that you should stay in an ascending-bid auction up to the exact moment at which the price reaches your true value. If we think of each bidder i's "drop-out price" as her bid b_i, this says that people should use their true values as their bids.

Moreover, with this definition of bids, the rule for determining the outcome of an ascending-bid auction can be reformulated as follows. The person with the highest bid is the one who stays in the longest, thus winning the item, and she pays the price at which the second-to-last person dropped out — in other words, she pays the bid of this second-to-last person. Thus, the item goes to the highest bidder at a price equal to the second-highest bid. This is precisely the rule used in the sealed-bid second-price auction, with the difference being that the ascending-bid auction involves real-time interaction between the buyers and seller, while the sealed-bid version takes place purely through sealed bids that the seller opens and evaluates. But the close similarity in rules helps to motivate the initially counter-intuitive pricing rule for the second-price auction: it can be viewed as a simulation, using sealed bids, of an ascending-bid auction. Moreover, the fact that bidders want to remain in an ascending-bid auction up to exactly the point at which their true value is reached provides the intuition for what will be our main result in the next section: after formulating the sealed-bid second-price auction in terms of game theory, we will find that bidding one's true value is a dominant strategy.

[1] It's conceptually simplest to think of three things happening simultaneously at the end of an ascending-bid auction: (i) the second-to-last bidder drops out; (ii) the last remaining bidder sees that she is alone and stops agreeing to any higher prices; and (iii) the seller awards the item to this last remaining bidder at the current price. Of course, in practice we might well expect that there is some very small increment by which the bid is raised in each step, and that the last remaining bidder actually wins only after one more raising of the bid by this tiny increment. But keeping track of this small increment makes for a more cumbersome analysis without changing the underlying ideas, and so we will assume that the auction ends at precisely the moment when the second-highest bidder drops out.

Comparing Auction Formats. In the next two sections we will consider the two main formats for sealed-bid auctions in more detail. Before doing this, it's worth making two points. First, the discussion in this section shows that when we analyze bidder behavior in sealed-bid auctions, we're also learning about their interactive analogues — with the descending-bid auction as the analogue of the sealed-bid first-price auction, and the ascending-bid auction as the analogue of the sealed-bid second-price auction.

Second, a purely superficial comparison of the first-price and second-price sealed-bid auctions might suggest that the seller would get more money for the item if he ran a first-price auction: after all, he'll get paid the highest bid rather than the second-highest bid. It may seem strange that in a second-price auction, the seller is intentionally undercharging the bidders. But such reasoning ignores one of the main messages from our study of game theory — that when you make up rules to govern people's behavior, you have to assume that they'll adapt their behavior in light of the rules. Here, the point is that bidders in a first-price auction will tend to bid *lower* than they do in a second-price auction, and in fact this lowering of bids will tend to offset what would otherwise look like a difference in the size of the winning bid. This consideration will come up as a central issue at various points later in the chapter.

9.4 Second-Price Auctions

The sealed-bid second-price auction is particularly interesting, and there are a number of examples of it in widespread use. The auction form used on eBay is essentially a second-price auction. The pricing mechanism that search engines use to sell keyword-based advertising is a generalization of the second-price auction, as we will see in Chapter 15. One of the most important results in auction theory is the fact we mentioned toward the end of the previous section: with independent, private values, bidding your true value is a dominant strategy in a second price sealed-bid auction. That is, the best choice of bid is exactly what the object is worth to you.

Formulating the Second-Price Auction as a Game. To see why this is true, we set things up using the language of game theory, defining the auction in terms of players, strategies, and payoffs. The bidders will correspond to the players. Let v_i be bidder i's true value for the object. Bidder i's strategy is an amount b_i to bid as a function of her true value v_i. In a second-price sealed-bid auction, the payoff to bidder i with value v_i and bid b_i is defined as follows.

> *If b_i is not the winning bid, then the payoff to i is 0. If b_i is the winning bid, and some other b_j is the second-place bid, then the payoff to i is $v_i - b_j$.*

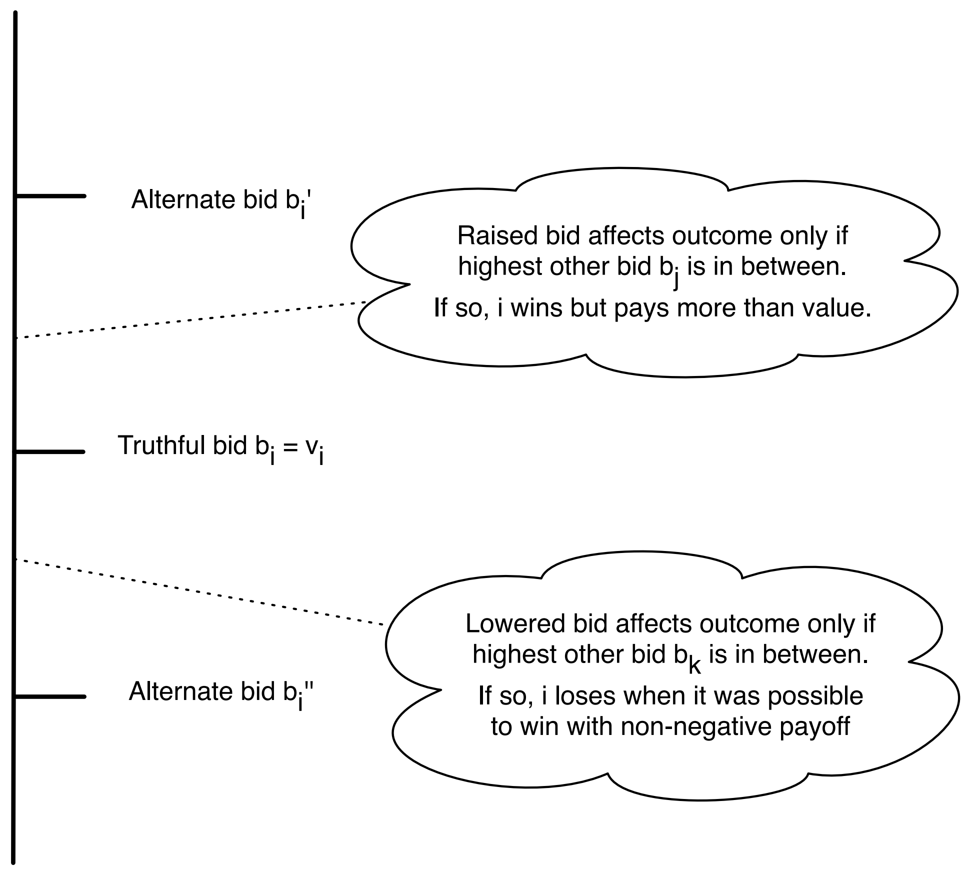

Figure 9.1: If bidder i deviates from a truthful bid in a second-price auction, the payoff is only affected if the change in bid changes the win/loss outcome.

To make this completely well-defined, we need to handle the possibility of ties: what do we do if two people submit the same bid, and it's tied for the largest? One way to handle this is to assume that there is a fixed ordering on the bidders that is agreed on in advance, and if a set of bidders ties for the numerically largest bid, then the winning bid is the one submitted by the bidder in this set that comes first in this order. Our formulation of the payoffs works with this more refined definition of "winning bid" and "second-place bid." (And note that in the case of a tie, the winning bidder receives the item but pays the full value of her own bid, for a payoff of zero, since in the event of a tie the first-place and second-place bids are equal.)

There is one further point worth noting about our formulation of auctions in the language of game theory. When we defined games in Chapter 6, we assumed that each player knew the payoffs of all players in the game. Here this isn't the case, since the bidders don't know each other's values, and so strictly speaking we need to use a slight generalization of the notions from Chapter 6 to handle this lack of knowledge. For our analysis here, however, since we are focusing on dominant strategies in which a player has an optimal strategy regardless of the other players' behavior, we will be able to disregard this subtlety.

Truthful Bidding in Second-Price Auctions. The precise statement of our claim about second-price auctions is as follows.

> *Claim: In a sealed-bid second-price auction, it is a dominant strategy for each bidder i to choose a bid $b_i = v_i$.*

To prove this claim, we need to show that if bidder i bids $b_i = v_i$, then no deviation from this bid would improve her payoff, regardless of what strategy everyone else is using. There are two cases to consider: deviations in which i raises her bid, and deviations in which i lowers her bid. The key point in both cases is that the value of i's bid only affects whether i wins or loses, but never affects how much i pays in the event that she wins — the amount paid is determined entirely by the other bids, and in particular by the largest among the other bids. Since all other bids remain the same when i changes her bid, a change to i's bid only affects her payoff if it changes her win/loss outcome. This argument is summarized in Figure 9.1.

With this in mind, let's consider the two cases. First, suppose that instead of bidding v_i, bidder i chooses a bid $b'_i > v_i$. This only affects bidder i's payoff if i would lose with bid v_i but would win with bid b'_i. In order for this to happen, the highest other bid b_j must be between b_i and b'_i. In this case, the payoff to i from deviating would be at most $v_i - b_j \leq 0$, and so this deviation to bid b'_i does not improve i's payoff.

Next, suppose that instead of bidding v_i, bidder i chooses a bid $b''_i < v_i$. This only affects bidder i's payoff if i would win with bid v_i but would lose with bid b''_i. So before deviating, v_i was the winning bid, and the second-place bid b_k was between v_i and b''_i. In this case, i's payoff before deviating was $v_i - b_k \geq 0$, and after deviating it is 0 (since i loses), so again this deviation does not improve i's payoff.

This completes the argument that truthful bidding is a dominant strategy in a sealed-bid second-price auction. The heart of the argument is the fact noted at the outset: in a second-price auction, your bid determines whether you win or lose, but not how much you pay in the event that you win. Therefore, you need to evaluate changes to your bid in light of this. This also further highlights the parallels to the ascending-bid auction. There too, the analogue of your bid — i.e. the point up to which you're willing to stay in the auction — determines whether you'll stay in long enough to win; but the amount you pay in the event that you win is determined by the point at which the second-place bidder drops out.

The fact that truthfulness is a dominant strategy also makes second-price auctions conceptually very clean. Because truthful bidding is a dominant strategy, it is the best thing to do regardless of what the other bidders are doing. So in a second-price auction, it makes sense to bid your true value even if other bidders are overbidding, underbidding, colluding, or behaving in other unpredictable ways. In other words, truthful bidding is a good idea even if the competing bidders in the auction don't know that they ought to be bidding truthfully as well.

We now turn to first-price auctions, where we'll find that the situation is much more complex. In particular, each bidder now has to reason about the behavior of her competitors in order to arrive at an optimal choice for her own bid.

9.5 First-Price Auctions and Other Formats

In a sealed-bid first-price auction, the value of your bid not only affects whether you win but also how much you pay. As a result, most of the reasoning from the previous section has to be redone, and the conclusions are now different.

To begin with, we can set up the first-price auction as a game in essentially the same way that we did for second-price auctions. As before, bidders are players, and each bidder's strategy is an amount to bid as a function of her true value. The payoff to bidder i with value v_i and bid b_i is simply the following.

> *If b_i is not the winning bid, then the payoff to i is 0. If b_i is the winning bid, then the payoff to i is $v_i - b_i$.*

The first thing we notice is that bidding your true value is no longer a dominant strategy. By bidding your true value, you would get a payoff of 0 if you lose (as usual), and you would also get a payoff of 0 if you win, since you'd pay exactly what it was worth to you.

As a result, the optimal way to bid in a first-price auction is to "shade" your bid slightly downward, so that if you win you will get a positive payoff. Determining how much to shade your bid involves balancing a trade-off between two opposing forces. If you bid too close to your true value, then your payoff won't be very large in the event that you win. But if you bid too far below your true value, so as to increase your payoff in the event of winning, then you reduce your chance of being the high bid and hence your chance of winning at all.

Finding the optimal trade-off between these two factors is a complex problem that depends on knowledge of the other bidders and their distribution of possible values. For example, it is intuitively natural that your bid should be higher — i.e. shaded less, closer to your true value — in a first-price auction with many competing bidders than in a first-price auction with only a few competing bidders (keeping other properties of the bidders the same). This is simply because with a large pool of other bidders, the highest competing bid is likely to be larger, and hence you need to bid higher to get above this and be the highest bid. We will discuss how to determine the optimal bid for a first-price auction in Section 9.7.

All-pay auctions. There are other sealed-bid auction formats that arise in different settings. One that initially seems counter-intuitive in its formulation is the *all-pay auction*: each bidder submits a bid; the highest bidder receives the item; and *all* bidders pay their bids, regardless of whether they win or lose. That is, the payoffs are now as follows.

> *If b_i is not the winning bid, then the payoff to i is $-b_i$. If b_i is the winning bid, then the payoff to i is $v_i - b_i$.*

Games with this type of payoff arise in a number of situations, usually where the notion of "bidding" is implicit. Political lobbying can be modeled in this way: each side must spend money on lobbying, but only the successful side receives anything of value for this expenditure. While it is not true that the side spending more on lobbying always wins, there is a clear analogy between the amount spent on lobbying and a bid, with all parties paying their bid regardless of whether they win or lose. One can picture similar considerations arising in settings such as design competitions, where competing architectural firms spend money on preliminary designs to try to win a contract from a client. This money must be spent before the client makes a decision.

The determination of an optimal bid in an all-pay auction shares a number of qualitative features with the reasoning in a first-price auction: in general you want to bid below your true value, and you must balance the trade-off between bidding high (increasing your probability of winning) and bidding low (decreasing your expenditure if you lose and increasing your payoff if you win). In general, the fact that everyone must pay in this auction format means that bids will typically be shaded much lower than in a first-price auction. The framework we develop for determining optimal bids in first-price auctions will also apply to all-pay auctions, as we will see in Section 9.7.

9.6 Common Values and The Winner's Curse

Thus far, we have assumed that bidders' values for the item being auctioned are independent: each bidder knows her own value for the item, and is not concerned with how much it is worth to anyone else. This makes sense in a lot of situations, but it clearly doesn't apply to a setting in which the bidders intend to resell the object. In this case, there is a common eventual value for the object — the amount it will generate on resale — but it is not necessarily known. Each bidder i may have some private information about the common value, leading to an estimate v_i of this value. Individual bidder estimates will typically be slightly wrong, and they will also typically not be independent of each other. One possible model for such estimates is to suppose that the true value is v, and that each bidder i's estimate v_i is defined by $v_i = v + x_i$, where x_i is a random number with a mean of 0, representing the error in i's estimate.

Auctions with common values introduce new sources of complexity. To see this, let's start by supposing that an item with a common value is sold using a second-price auction. Is it still a dominant strategy for bidder i to bid v_i? In fact, it's not. To get a sense for why this is, we can use the model with random errors $v + x_i$. Suppose there are many bidders, and that each bids her estimate of the true value. Then from the result of the auction, the winning bidder not only receives the object, she also learns something about her estimate of the common value — that it was the highest of all the estimates. So in particular, her estimate is more likely to be an over-estimate of the common value than an under-estimate. Moreover, with many bidders, the second-place bid — which is what she paid — is also likely to be an over-estimate. As a result she will likely lose money on the resale relative to what she paid.

This is known as the *winner's curse*, and it is a phenomenon that has a rich history in the study of auctions. Richard Thaler's review of this history [387] notes that the winner's curse appears to have been first articulated by researchers in the petroleum industry [95]. In this domain, firms bid on oil-drilling rights for tracts of land that have a common value, equal to the value of the oil contained in the tract. The winner's curse has also been studied in the context of competitive contract offers to baseball free agents [98] — with the unknown common value corresponding to the future performance of the baseball player being courted.[2]

Rational bidders should take the winner's curse into account in deciding on their bids: a bidder should bid her best estimate of the value of the object conditional on both her private estimate v_i and on winning the object at her bid. That is, it must be the case that at an optimal bid, it is better to win the object than not to win it. This means in a common-value auction, bidders will shade their bids downward even when the second-price format is used; with the first-price format, bids will be reduced even further. Determining the optimal bid is fairly complex, and we will not pursue the details of it here. It is also worth noting that in practice, the winner's curse can lead to outright losses on the part of the winning bidder [387], since in a large pool of bidders, anyone who in fact makes an error and overbids is more likely to be the winner of the auction.

[2]In these cases as well as others, one could argue that the model of common values is not entirely accurate. One oil company could in principle be more successful than another at extracting oil from a tract of land; and a baseball free agent may flourish if he joins one team but fail if he joins another. But common values are a reasonable approximation to both settings, as to any case where the purpose of bidding is to obtain an item that has some intrinsic but unknown future value. Moreover, the reasoning behind the winner's curse arises even when the item being auctioned has related but non-identical values to the different bidders.

9.7 Advanced Material: Bidding Strategies in First-Price and All-Pay Auctions

In the previous two sections we offered some intuition about the way to bid in first-price auctions and in all-pay auctions, but we did not derive optimal bids. We now develop models of bidder behavior under which we can derive equilibrium bidding strategies in these auctions. We then explore how optimal behavior varies depending on the number of bidders and on the distribution of values. Finally, we analyze how much revenue the seller can expect to obtain from various auctions. The analysis in this section will use elementary calculus and probability theory.

A. Equilibrium Bidding in First-Price Auctions

As the basis for the model, we want to capture a setting in which bidders know how many competitors they have, and they have partial information about their competitors' values for the item. However, they do not know their competitors' values exactly.

Let's start with a simple case first, and then move on to a more general formulation. In the simple case, suppose that there are two bidders, each with a private value that is independently and uniformly distributed between 0 and 1.[3] This information is common knowledge among the two bidders. A *strategy* for a bidder is a function $s(v) = b$ that maps her true value v to a non-negative bid b. We will make the following simple assumptions about the strategies the bidders are using:

(i) $s(\cdot)$ is a strictly increasing, differentiable function; so in particular, if two bidders have different values, then they will produce different bids.

(ii) $s(v) \leq v$ for all v: bidders can shade their bids down, but they will never bid above their true values. Notice that since bids are always non-negative, this also means that $s(0) = 0$.

These two assumptions permit a wide range of strategies. For example, the strategy of bidding your true value is represented by the function $s(v) = v$, while the strategy of shading your bid downward to by a factor of $c < 1$ times your true value is represented by $s(v) = cv$. More complex strategies such as $s(v) = v^2$ are also allowed, although we will see that in first-price auctions they are not optimal.

The two assumptions help us narrow the search for equilibrium strategies. The second of our assumptions only rules out strategies (based on overbidding) that are non-optimal.

[3]The fact that the 0 and 1 are the lowest and highest possible values is not crucial; by shifting and re-scaling these quantities, we could equally well consider values that are uniformly distributed between any other pair of endpoints.

The first assumption restricts the scope of possible equilibrium strategies, but it makes the analysis easier while still allowing us to study the important issues.

Finally, since the two bidders are identical in all ways except the actual value they draw from the distribution, we will narrow the search for equilibria in one further way: we will consider the case in which the two bidders follow the same strategy $s(\cdot)$.

Equilibrium with two bidders: The Revelation Principle. Let's consider what such an equilibrium strategy should look like. First, assumption (i) says that the bidder with the higher value will also produce the higher bid. If bidder i has a value of v_i, the probability that this is higher than the value of i's competitor in the interval $[0, 1]$ is exactly v_i. Therefore, i will win the auction with probability v_i. If i does win, i receives a payoff of $v_i - s(v_i)$. Putting all this together, we see that i's expected payoff is

$$g(v_i) = v_i(v_i - s(v_i)). \tag{9.1}$$

Now, what does it mean for $s(\cdot)$ to be an equilibrium strategy? It means that for each bidder i, there is no incentive for i to deviate from strategy $s(\cdot)$ if i's competitor is also using strategy $s(\cdot)$. It's not immediately clear how to analyze deviations to an arbitrary strategy satisfying assumptions (i) and (ii) above. Fortunately, there is an elegant device that lets us reason about deviations as follows: rather then actually switching to a different strategy, bidder i can implement her deviation by keeping the strategy $s(\cdot)$ but supplying a different "true value" to it.

Here is how this works. First, if i's competitor is also using strategy $s(\cdot)$, then i should never announce a bid above $s(1)$, since i can win with bid $s(1)$ and get a higher payoff with bid $s(1)$ than with any bid $b > s(1)$. So in any possible deviation by i, the bid she will actually report will lie between $s(0) = 0$ and $s(1)$. Therefore, for the purposes of the auction, she can simulate her deviation to an alternate strategy by first pretending that her true value is v_i' rather than v_i, and then applying the existing function $s(\cdot)$ to v_i' instead of v_i. This is a special case of a much broader idea known as the *Revelation Principle* [124, 207, 310]; for our purposes, we can think of it as saying that deviations in the bidding strategy function can instead be viewed as deviations in the "true value" that bidder i supplies to her current strategy $s(\cdot)$.

With this in mind, we can write the condition that i does not want to deviate from strategy $s(\cdot)$ as follows:

$$v_i(v_i - s(v_i)) \geq v(v_i - s(v)) \tag{9.2}$$

for all possible alternate "true values" v between 0 and 1 that bidder i might want to supply to the function $s(\cdot)$.

Is there a function that satisfies this property? In fact, it is not hard to check that $s(v) = v/2$ satisfies it. To see why, notice that with this choice of $s(\cdot)$, the left-hand side of Inequality (9.2) becomes $v_i(v_i - v_i/2) = v_i^2/2$ while the right-hand side becomes $v(v_i - v/2) = vv_i - v^2/2$. Collecting all the terms on the left, the inequality becomes simply

$$\frac{1}{2}(v^2 - 2vv_i + v_i^2) \geq 0,$$

which holds because the left-hand side is the square $\frac{1}{2}(v - v_i)^2$.

Thus, the conclusion in this case is quite simple to state. If two bidders know they are competing against each other, and know that each has a private value drawn uniformly at random from the interval $[0, 1]$, then it is an equilibrium for each to shade their bid down by a factor of 2. Bidding half your true value is optimal behavior if the other bidder is doing this as well.

Note that unlike the case of the second-price auction, we have not identified a dominant strategy, only an equilibrium. In solving for a bidder's optimal strategy we used each bidder's expectation about her competitor's bidding strategy. In an equilibrium, these expectations are correct. But if other bidders for some reason use non-equilibrium strategies, then any bidder should optimally respond and potentially also play some other bidding strategy.

Deriving the two-bidder equilibrium. In our discussion of the equilibrium $s(v) = v/2$, we initially conjectured the form of the function $s(\cdot)$, and then checked that it satisfied Inequality (9.2). But this approach does not suggest how to discover a function $s(\cdot)$ to use as a conjecture.

An alternate approach is to derive $s(\cdot)$ directly by reasoning about the condition in Inequality (9.2). Here is how we can do this. In order for $s(\cdot)$ to satisfy Inequality (9.2), it must have the property that for any true value v_i, the expected payoff function $g(v) = v(v_i - s(v))$ is maximized by setting $v = v_i$. Therefore, v_i should satisfy $g'(v_i) = 0$, where g' is the first derivative of $g(\cdot)$ with respect to v. Since

$$g'(v) = v_i - s(v) - vs'(v)$$

by the Product Rule for derivatives, we see that $s(\cdot)$ must solve the differential equation

$$s'(v_i) = 1 - \frac{s(v_i)}{v_i}$$

for all v_i in the interval $[0, 1]$. This differential equation is solved by the function $s(v_i) = v_i/2$.

Equilibrium with Many Bidders. Now let's suppose that there are n bidders, where n can be larger than two. To start with, we'll continue to assume that each bidder i draws her true value v_i independently and uniformly at random from the interval between 0 and 1.

Much of the reasoning for the case of two bidders still works here, although the basic formula for the expected payoff changes. Specifically, assumption (i) still implies that the bidder with the highest true value will produce the highest bid and hence win the auction. For a given bidder i with true value v_i, what is the probability that her bid is the highest? This requires each other bidder to have a value below v_i; since the values are chosen independently, this event has a probability of v_i^{n-1}. Therefore, bidder i's expected payoff is

$$G(v_i) = v_i^{n-1}(v_i - s(v_i)). \tag{9.3}$$

The condition for $s(\cdot)$ to be an equilibrium strategy remains the same as it was in the case of two bidders. Using the Revelation Principle, we view a deviation from the bidding strategy as supplying a "fake" value v to the function $s(\cdot)$; given this, we require that the true value v_i produces an expected payoff at least as high as the payoff from any deviation:

$$v_i^{n-1}(v_i - s(v_i)) \geq v^{n-1}(v_i - s(v)) \tag{9.4}$$

for all v between 0 and 1.

From this, we can derive the form of the bidding function $s(\cdot)$ using the differential-equation approach that worked for two bidders. The expected payoff function $G(v) = v^{n-1}(v_i - s(v))$ must be maximized by setting $v = v_i$. Setting the derivative $G'(v_i) = 0$ and applying the Product Rule to differentiate G, we get

$$(n-1)v^{n-2}v_i - (n-1)v^{n-2}s(v_i) - v_i^{n-1}s'(v_i) = 0$$

for all v_i between 0 and 1. Dividing through by $(n-1)v^{n-2}$ and solving for $s'(v_i)$, we get the equivalent but typographically simpler equation

$$s'(v_i) = (n-1)\left(1 - \frac{s(v_i)}{v_i}\right) \tag{9.5}$$

for all v_i between 0 and 1. This differential equation is solved by the function

$$s(v_i) = \left(\frac{n-1}{n}\right)v_i.$$

So if each bidder shades her bid down by a factor of $(n-1)/n$, then this is optimal behavior given what everyone else is doing. Notice that when $n = 2$ this is our two-bidder strategy. The form of this strategy highlights an important principle that we discussed in Section 9.5 about strategic bidding in first-price auctions: as the number of bidders increases, you generally have to bid more "aggressively," shading your bid down less, in order to win. For the simple case of values drawn independently from the uniform distribution, our analysis here quantifies exactly how this increased aggressiveness should depend on the number of bidders n.

General Distributions. In addition to considering larger numbers of bidders, we can also relax the assumption that bidders' values are drawn from the uniform distribution on an interval.

Suppose that each bidder has her value drawn from a probability distribution over the non-negative real numbers. We can represent the probability distribution by its *cumulative distribution function* $F(\cdot)$: for any x, the value $F(x)$ is the probability that a number drawn from the distribution is at most x. We will assume that F is a differentiable function.

Most of the earlier analysis continues to hold at a general level. The probability that a bidder i with true value v_i wins the auction is the probability that no other bidder has a larger value, so it is equal to $F(v_i)^{n-1}$. Therefore, the expected payoff to v_i is

$$F(v_i)^{n-1}(v_i - s(v_i)).$$

Then, the requirement that bidder i does not want to deviate from this strategy becomes

$$F(v_i)^{n-1}(v_i - s(v_i)) \geq F(v)^{n-1}(v_i - s(v)) \tag{9.6}$$

for all v between 0 and 1.

Finally, this equilibrium condition can be used to write a differential equation just as before, using the fact that the function of v on the right-hand side of Inequality (9.6) should be maximized when $v = v_i$. We apply the Product Rule, and also the Chain Rule for derivatives, keeping in mind that the derivative of the cumulative distribution function $F(\cdot)$ is the probability density function $f(\cdot)$ for the distribution. Proceeding by analogy with the analysis for the uniform distribution, we get the differential equation

$$s'(v_i) = (n-1)\left(\frac{f(v_i)v_i - f(v_i)s(v_i)}{F(v_i)}\right). \tag{9.7}$$

Notice that for the uniform distribution on the interval $[0, 1]$, the cumulative distribution function is $F(v) = v$ and the density is $f(v) = 1$, which applied to Equation (9.7) gives us back Equation (9.5).

Finding an explicit solution to Equation (9.7) isn't possible unless we have an explicit form for the distribution of values, but it provides a framework for taking arbitrary distributions and solving for equilibrium bidding strategies.

B. Seller Revenue

Now that we've analyzed bidding strategies for first-price auctions, we can return to an issue that came up at the end of Section 9.3: how to compare the revenue a seller should expect to make in first-price and second-price auctions.

There are two competing forces at work here. On the one hand, in a second-price auction, the seller explicitly commits to collecting less money, since he only charges the second-highest bid. On the other hand, in a first-price auction, the bidders reduce their bids, which also reduces what the seller can collect.

To understand how these opposing factors trade off against each other, suppose we have n bidders with values drawn independently from the uniform distribution on the interval $[0, 1]$. Since the seller's revenue will be based on the values of the highest and second-highest bids, which in turn depend on the highest and second-highest values, we need to know the expectations of these quantities.[4] Computing these expectations is complicated, but the form of the answer is very simple. Here is the basic statement:

> *Suppose n numbers are drawn independently from the uniform distribution on the interval $[0, 1]$ and then sorted from smallest to largest. The expected value of the number in the k^{th} position on this list is $\frac{k}{n+1}$.*

Now, if the seller runs a second-price auction, and the bidders follow their dominant strategies and bid truthfully, the seller's expected revenue will be the expectation of the second-highest value. Since this will be the value in position $n - 1$ in the sorted order of the n random values from smallest to largest, the expected value is $(n - 1)/(n + 1)$, by the formula just described. On the other hand, if the seller runs a first-price auction, then in equilibrium we expect the winning bidder to submit a bid that is $(n - 1)/n$ times her true value. Her true value has an expectation of $n/(n + 1)$ (since it is the largest of n numbers drawn independently from the unit interval), and so the seller's expected revenue is

$$\left(\frac{n-1}{n}\right)\left(\frac{n}{n+1}\right) = \frac{n-1}{n+1}.$$

The two auctions provide exactly the same expected revenue to the seller!

Revenue Equivalence. As far as seller revenue is concerned, this calculation is in a sense the tip of the iceberg: it is a reflection of a much broader and deeper principle known in the auction literature as *revenue equivalence* [256, 288, 311]. Roughly speaking, revenue equivalence asserts that a seller's revenue will be the same across a broad class of auctions and arbitrary independent distributions of bidder values, when bidders follow equilibrium strategies. A formalization and proof of the revenue equivalence principle can be found in [256].

From the discussion here, it is easy to see how the ability to commit to a selling mechanism is valuable for a seller. Consider, for example, a seller using a second-price auction. If the bidders bid truthfully and the seller does not sell the object as promised, then the seller knows the bidders' values and can bargain with them from this advantaged position. At worst, the seller should be able to sell the object to the bidder with the highest value at a

[4] In the language of probability theory, these are known as the expectations of the *order statistics*.

price equal to the second highest value. (The bidder with the highest value knows that if she turns down the trade at this price, then the bidder with the second-highest value will take it.) But the seller may be able to do better than this in the negotiation, and so overall the bidders lose relative to the originally promised second-price auction. If bidders suspect that this scenario may occur with some probability, then they may no longer find it optimal to bid truthfully in the auction, and so it is not clear what the seller receives.

Reserve Prices. In our discussion of how a seller should choose an auction format, we have implicitly assumed that the seller must sell the object. Let's briefly consider how the seller's expected revenue changes if he has the option of holding onto the item and choosing not to sell it. To be able to reason about the seller's payoff in the event that this happens, let's assume that the seller values the item at $u \geq 0$, which is thus the payoff he gets from keeping the item rather than selling it.

It's clear that if $u > 0$, then the seller should not use a simple first-price or second-price auction. In either case, the winning bid might be less than u, and the seller would not want to sell the object. If the seller refuses to sell after having specified a first-price or second-price auction, then we are back in the case of a seller who might break his initial commitment to the format.

Instead, it is better for the seller to announce a *reserve price* of r before running the auction. With a reserve price, the item is sold to the highest bidder *if* the highest bid is above r; otherwise, the item is not sold. In a first-price auction with a reserve price, the winning bidder (if there is one) still pays her bid. In a second-price auction with a reserve price, the winning bidder (if there is one) pays the maximum of the second-place bid and the reserve price r. As we will see, it is in fact useful for the seller to declare a reserve price even if his value for the item is $u = 0$.

Let's consider how to reason about the optimal value for the reserve price in the case of a second-price auction. First, it is not hard to go back over the argument that truthful bidding is a dominant strategy in second-price auctions and check that it still holds in the presence of a reserve price. Essentially, it is as if the seller were another "simulated" bidder who always bids r; and since truthful bidding is optimal regardless of how other bidders behave, the presence of this additional simulated bidder has no effect on how any of the real bidders should behave.

Now, what value should the seller choose for the reserve price? If the item is worth u to the seller, then clearly he should set $r \geq u$. But in fact the reserve price that maximizes the seller's expected revenue is strictly greater than u. To see why this is true, let's first consider a very simple case: a second-price auction with a single bidder, whose value is uniformly distributed on $[0, 1]$, and a seller whose value for the item is $u = 0$. With only one bidder, the second-price auction with no reserve price will sell the item to the bidder at a price of

0. On the other hand, suppose the seller sets a reserve price of $r > 0$. In this case, with probability $1 - r$, the bidder's value is above r, and the object will be sold to the bidder at a price of r. With probability r, the bidder's value is below r, and so the seller keeps the item, receiving a payoff of $u = 0$. Therefore, the seller's expected revenue is $r(1 - r)$, and this is maximized at $r = 1/2$. If the seller's value u is greater than zero, then his expected payoff is $r(1 - r) + ru$ (since he receives a payoff of u when the item is not sold), and this is maximized by setting $r = (1 + u)/2$. So with a single bidder, the optimal reserve price is halfway between the value of the object to the seller and the maximum possible bidder value. With more intricate analyses, one can similarly determine the optimal reserve price for a second-price auction with multiple bidders, as well as for a first-price auction with equilibrium bidding strategies of the form we derived earlier.

C. Equilibrium Bidding in All-Pay Auctions

The style of analysis we've been using for first-price auctions can be adapted without much difficulty to other formats as well. Here we will show how this works for the analysis of all-pay auctions: recall from Section 9.5 that this is an auction format — designed to model activities such as lobbying — where the highest bidder wins the item but everyone pays their bid.

We will keep the general framework we used for first-price auctions earlier in this section, with n bidders, each with a value drawn independently and uniformly at random from between 0 and 1. As before, we want to find a function $s(\cdot)$ mapping values to bids, so that using $s(\cdot)$ is optimal if all other bidders are using it.

With an all-pay auction, the expected payoff for bidder i has a negative term if i does not win. The formula is now

$$v_i^{n-1}(v_i - s(v_i)) + (1 - v_i^{n-1})(-s(v_i)),$$

where the first term corresponds to the payoff in the event that i wins, and the second term corresponds to the payoff in the event that i loses. As before, we can think of a deviation from this bidding strategy as supplying a fake value v to the function $s(\cdot)$; so if $s(\cdot)$ is an equilibrium choice of strategies by the bidders, then

$$v_i^{n-1}(v_i - s(v_i)) + (1 - v_i^{n-1})(-s(v_i)) \geq v^{n-1}(v_i - s(v)) + (1 - v^{n-1})(-s(v)) \qquad (9.8)$$

for all v in the interval $[0, 1]$.

Notice that the expected payoff consists of a fixed cost $s(v)$ that is paid regardless of the win/loss outcome, plus a value of v_i in the event that i wins. Canceling the common terms in Inequality (9.8), we can rewrite it as

$$v_i^n - s(v_i) \geq v^{n-1}v_i - s(v). \qquad (9.9)$$

for all v in the interval $[0,1]$. Now, writing the right-hand side as a function $g(v) = v^{n-1}v_i - s(v)$, we can view Inequality (9.9) as requiring that $v = v_i$ maximizes the function $g(\cdot)$. The resulting equation $g'(v_i) = 0$ then gives us a differential equation that specifies $s(\cdot)$ quite simply:

$$s'(v_i) = (n-1)v_i^{n-1},$$

and hence $s(v) = \left(\frac{n-1}{n}\right) v_i^n$

Since $v_i < 1$, raising it to the n^{th} power (as specified by the function $s(\cdot)$) reduces it exponentially in the number of bidders. This shows that bidders will shade their bids downward significantly as the number of bidders in an all-pay auction increases.

We can also work out the seller's expected revenue. The seller collects money from everyone in an all-pay auction; on the other hand, the bidders all submit low bids. The expected value of a single bidder's contribution to seller revenue is simply

$$\int_0^1 s(v)\ dv = \left(\frac{n-1}{n}\right) \int_0^1 v^n\ dv = \left(\frac{n-1}{n}\right)\left(\frac{1}{n+1}\right).$$

Since the seller collects this much in expectation from each bidder, the seller's overall expected revenue is

$$n\left(\frac{n-1}{n}\right)\left(\frac{1}{n+1}\right) = \frac{n-1}{n+1}.$$

This is exactly the same as the seller's expected revenue in the first-price and second-price auctions with the same assumptions about bidder values. Again, this is a reflection of the much broader revenue equivalence principle [256, 288], which includes all-pay auctions in the general set of auction formats it covers.

9.8 Exercises

1. In this question we will consider an auction in which there is one seller who wants to sell one unit of a good and a group of bidders who are each interested in purchasing the good. The seller will run a sealed-bid, second-price auction. Your firm will bid in the auction, but it does not know for sure how many other bidders will participate in the auction. There will be either two or three other bidders in addition to your firm. All bidders have independent, private values for the good. Your firm's value for the good is c. What bid should your firm submit, and how does it depend on the number of other bidders who show up? Give a brief (1-3 sentence) explanation for your answer.

2. In this problem we will ask how the number of bidders in a second-price, sealed-bid auction affects how much the seller can expect to receive for his object. Assume that there are two bidders who have independent, private values v_i which are either 1 or 3.

For each bidder, the probabilities of 1 and 3 are both 1/2. (If there is a tie at a bid of x for the highest bid the winner is selected at random from among the highest bidders and the price is x.)

(a) Show that the seller's expected revenue is 6/4.

(b) Now let's suppose that there are three bidders who have independent, private values v_i which are either 1 or 3. For each bidder, the probabilities of 1 and 3 are both 1/2. What is the seller's expected revenue in this case?

(c) Briefly explain why changing the number of bidders affects the seller's expected revenue.

3. In this problem we will ask how much a seller can expect to receive for his object in a second-price, sealed-bid auction. Assume that all bidders have independent, private values v_i which are either 0 or 1. The probability of 0 and 1 are both 1/2.

   (a) Suppose there are two bidders. Then there are four possible pairs of their values (v_1, v_2): $(0,0), (1,0), (0,1)$, and $(1,1)$. Each pair of values has probability 1/4. Show that the seller's expected revenue is 1/4. (Assume that if there is a tie at a bid of x for the highest bid the winner is selected at random from among the highest bidders and the price is x.)

   (b) What is the seller's expected revenue if there are three bidders?

   (c) This suggests a conjecture that as the number of bidders increases the seller's expected revenue also increases. In the example we are considering the seller's expected revenue actually converges to 1 as the number of bidders grows. Explain why this should occur. You do not need to write a proof; an intuitive explanation is fine.

4. A seller will run a second-price, sealed-bid auction for an object. There are two bidders, a and b, who have independent, private values v_i which are either 0 or 1. For both bidders the probabilities of $v_i = 0$ and $v_i = 1$ are each 1/2. Both bidders understand the auction, but bidder b sometimes makes a mistake about his value for the object. Half of the time his value is 1 and he is aware that it is 1; the other half of the time his value is 0 but occasionally he mistakenly believes that his value is 1. Let's suppose that when b's value is 0 he acts as if it is 1 with probability $\frac{1}{2}$ and as if it is 0 with probability $\frac{1}{2}$. So in effect bidder b sees value 0 with probability $\frac{1}{4}$ and value 1 with probability $\frac{3}{4}$. Bidder a never makes mistakes about his value for the object, but he is aware of the mistakes that bidder b makes. Both bidders bid optimally given their perceptions of the value of the object. Assume that if there is a tie at a bid of x for the highest bid the winner is selected at random from among the highest bidders and the price is x.

(a) Is bidding his true value still a dominant strategy for bidder a? Explain briefly

(b) What is the seller's expected revenue? Explain briefly.

5. Consider a second-price, sealed-bid auction with one seller who has one unit of the object which he values at s and two buyers $1, 2$ who have values of v_1 and v_2 for the object. The values s, v_1, v_2 are all independent, private values. Suppose that both buyers know that the seller will submit his own sealed bid of s, but they do not know the value of s. Is it optimal for the buyers to bid truthfully; that is should they each bid their true value? Give an explanation for your answer.

6. In this question we will consider the effect of collusion between bidders in a second-price, sealed-bid auction. There is one seller who will sell one object using a second-price sealed-bid auction. The bidders have independent, private values drawn from a distribution on $[0, 1]$. If a bidder with value v gets the object at price p, his payoff is $v - p$; if a bidder does not get the object his payoff is 0. We will consider the possibility of collusion between two bidders who know each others' value for the object. Suppose that the objective of these two colluding bidders is to choose their two bids as to maximize the sum of their payoffs. The bidders can submit any bids they like as long as the bids are in $[0, 1]$.

   (a) Let's first consider the case in which there are only two bidders. What two bids should they submit? Explain.

   (b) Now suppose that there is a third bidder who is not part of the collusion. Does the existence of this bidder change the optimal bids for the two bidders who are colluding? Explain.

7. A seller announces that he will sell a case of rare wine using a sealed-bid, second-price auction. A group of I individuals plan to bid on this case of wine. Each bidder is interested in the wine for his or her personal consumption; the bidders' consumption values for the wine may differ, but they don't plan to resell the wine. So we will view their values for the wine as independent, private values (as in Chapter 9). You are one of these bidders; in particular, you are bidder number i and your value for the wine is v_i.

   How should you bid in each of the following situations? In each case, provide an explanation for your answer; a formal proof is not necessary.

   (a) You know that a group of the bidders will collude on bids. This group will chose one bidder to submit a "real bid" of v and the others will all submit bids of 0. You are not a member of this collusive group and you cannot collude with any other bidder.

(b) You, and all of the other bidders, have just learned that this seller will collect bids, but won't actually sell the wine according to the rules of a second-price auction. Instead, after collecting the bids the seller will tell all of the bidders that some other fictional bidder actually submitted the highest bid and so won the auction. This bidder, of course, doesn't exist so the seller will still have the wine after the auction is over. The seller plans to privately contact the highest actual bidder and tell him or her that the fictional high bidder defaulted (he didn't buy the wine after all) and that this bidder can buy the wine for the price he or she bid in the auction. You cannot collude with any bidder. [You do not need to derive an optimal bidding strategy. It is enough to explain whether your bid would differ from your value and if so in what direction.]

8. In this problem we will ask how irrational behavior on the part of one bidder affects optimal behavior for the other bidders in an auction. In this auction the seller has one unit of the good which will be sold using a second-price, sealed-bid auction. Assume that there are three bidders who have independent, private values for the good, v_1, v_2 v_3, which are uniformly distributed on the interval $[0, 1]$.

   (a) Suppose first that all bidders behave rationally; that is they submit optimal bids. Which bidder (in terms of values) wins the auction and how much does this bidder pay (again in terms of the bidder's values)?

   (b) Suppose now that bidder 3 irrationally bids more than his true value for the object; in particular, bidder 3's bid is $(v_3 + 1)/2$. All other bidders know that bidder 3 is irrational in this way, although they do not know bidder 3's actual value for the object. How does this affect the behavior of the other bidders?

   (c) What effect does bidder 3's irrational behavior have on the expected payoffs of bidder 1? Here the expectation is over the values of v_2 and v_3 which bidder 1 does not know. You do not need to provide an explicit solution or write a proof for your answer; an intuitive explanation of the effect is fine. [Remember a bidder's payoff is the bidder's value for the object minus the price, if the bidder wins the auction; or 0, if the bidder does not win the auction.]

9. In this problem we will ask how much a seller can expect to receive for his object in a second-price, sealed-bid auction. Assume that there are two bidders who have independent, private values v_i which are either 1 or 2. For each bidder, the probabilities of $v_i = 1$ and $v_i = 2$ are each $1/2$. Assume that if there is a tie at a bid of x for the highest bid the winner is selected at random from among the highest bidders and the price is x. We also assume that the value of the object to the seller is 0.

   (a) Show that the seller's expected revenue is $5/4$.

(b) Now let's suppose that the seller sets a reserve price of R with $1 < R < 2$: that is, the object is sold to the highest bidder if her bid is at least R, and the price this bidder pays is the maximum of the second highest bid and R. If no bid is at least R, then the object is not sold, and the seller receives 0 revenue. Suppose that all bidders know R. What is the seller's expected revenue as a function of R?

(c) Using the previous part, show that a seller who wants to maximize expected revenue would never set a reserve price, R, that is more than 1 and less than 1.5.

10. In this problem we will examine a second-price, sealed-bid auction. Assume that there are two bidders who have independent, private values v_i which are either 1 or 7. For each bidder, the probabilities of $v_i = 1$ and $v_i = 7$ are each 1/2. So there are four possible pairs of the bidders' values (v_1, v_2): $(1,1), (1,7), (7,1),$ and $(7,7)$. Each pair of values has probability 1/4.

Assume that if there is a tie at a bid of x for the highest bid the winner is selected at random from among the highest bidders and the price is x.

(a) For each pair of values, what bid will each bidder submit, what price will the winning bidder pay, and how much profit (the difference between the winning bidder's value and price he pays) will the winning bidder earn?

(b) Now let's examine how much revenue the seller can expect to earn and how much profit the bidders can expect to make in the second price auction. Both revenue and profit depend on the values, so let's calculate the average of each of these numbers across all four of the possible pairs of values. [Note that in doing this we are computing each bidder's expected profit before the bidder knows his value for the object.] What is the seller's expected revenue in the second price auction? What is the expected profit for each bidder?

(c) The seller now decides to charge an entry fee of 1. Any bidder who wants to participate in the auction must pay this fee to the seller before bidding begins and, in fact, this fee is imposed before each bidder knows his or her own value for the object. The bidders know only the distribution of values and that anyone who pays the fee will be allowed to participate in a second price auction for the object. This adds a new first stage to the game in which bidders decide simultaneously whether to pay the fee and enter the auction, or to not pay the fee and stay out of the auction. This first stage is then followed by a second stage in which anyone who pays the fee participates in the auction. We will assume that after the first stage is over both potential bidders learn their own value for the object (but not the other potential bidder's value for the object) and that they both learn whether or not the other potential bidder decided to enter the auction.

Let's assume that any potential bidder who does not participate in the auction has a profit of 0, if no one chooses to participate then the seller keeps the object and does not run an auction, if only one bidder chooses to participate in the auction then the seller runs a second price auction with only this one bidder (and treats the second highest bid as 0), and finally if both bidders participate the second price auction is the one you solved in part (a).

Is there an equilibrium in which each bidder pays the fee and participates in the auction? Give an explanation for your answer.

11. In this question we will examine a second-price, sealed-bid auction for a single item. We'll consider a case in which true values for the item may differ across bidders, and it requires extensive research by a bidder to determine her own true value for an item — maybe this is because the bidder needs to determine her ability to extract value from the item after purchasing it (and this ability may differ from bidder to bidder).

    There are three bidders. Bidders 1 and 2 have values v_1 and v_2, each of which is a random number independently and uniformly distributed on the interval $[0, 1]$. Through having performed the requisite level of research, bidders 1 and 2 know their own values for the item, v_1 and v_2, respectively, but they do not know each other's value for item.

    Bidder 3 has not performed enough research to know his own true value for the item. He does know that he and bidder 2 are extremely similar, and therefore that his true value v_3 is exactly equal to the true value v_2 of bidder 2. The problem is that bidder 3 does not know this value v_2 (nor does he know v_1).

    (a) How should bidder 1 bid in this auction? How should bidder 2 bid?

    (b) How should bidder 3 behave in this auction? Provide an explanation for your answer; a formal proof is not necessary.