ECON-C1100 Mathematics for Economists

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Lecture 1

Welcome!

Course organization

Course organization

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Course organization

- 22 lectures + 8 exercise sessions
- Course page: https://mycourses.aalto.fi/course/view.php?id=37338
- The **exercise sessions** will be conducted by the TA
- Prerequisites: 30A01100 and 30A03000. Students are assumed to be familiar with the content of Chapters 1-5 of the textbook.

Course organization

- *Lecture slides* will be distributed but they are not meant to be self-contained. For each topic covered in class, you are required to read the corresponding parts in the textbook
- *Problem sets (exercises)* will be distributed during the course: 3 in the first period, and 3 in the second period. Students will be required to upload their answers to MyCourses. Working through the exercises is indispensable, one cannot learn mathematics without doing the exercises. Solutions to the problem sets will be provided and some of the exercises in the problem sets will be discussed in the exercise sessions.

Course organization

Course overview and goals: The aim of the course is to provide students with a thorough understanding of the main mathematical concepts and tools used in Economics. We will study:

- **linear algebra**: we will focus on how to solve systems of linear equations;
- **calculus of several variables**: we will analyze functions of several variables and learn how to apply differential calculus to them;
- **optimization**: we will learn how to find the points in the domain of a function that maximize or minimize the function's value, with or without constraints;
- **dynamical systems**: we will focus on how to solve difference and differential equations.

Course organization

List of topics (subject to change):

- **First Period**:
  - Lectures 1-4: Linear Algebra: Systems of linear equations, matrix algebra, linear independence (Chapters 6-11)
  - Lectures 5-8: Calculus of Several Variables: Functions of several variables, differential calculus, implicit functions (Chapters 12-15)
  - Lectures 9-11: Convexity and Unconstrained Optimization: Convex, concave and quasiconcave functions, quadratic forms, unconstrained optimization (Chapters 16-17, 21)
  - Lecture 12: Review
- **Second Period**:
  - Lectures 13-15: Constrained Optimization (Chapters 18-19)
  - Lectures 16-17: Difference equations (Chapters 23)
  - Lecture 18-21: Ordinary differential equations (Chapters 24-25)
  - Lecture 22: Review

Textbook: Carl P. Simon and Lawrence Blume, *Mathematics for Economists*, Norton, 1994

Course requirements and grading

- The final grade will be determined by:
  - (20%) problem sets;
  - (80%) exam.
- To meet the exam requirement, you can:
  - take two partial exams during the course (Midterm on 21.10.2022 and Final on 15.12.2022). The midterm will be on the material of the first period, and the final will cover the material of the second period. Each partial exam will be given equal weight.
  - Alternatively, take an exam which covers the entire class material after the end of the course (6.2.2023 or 27.3.2023).

The points earned with the problem sets will be valid for the entire academic year. The problem sets and exam questions will be in English. You can give answers in English or Finnish.

Introduction

Introduction

- Mathematics has become a common language for economists

- Advantages of using math:
  - conciseness, precision, and clarity
  - it gives us powerful conceptual tools that we can use to reason about complex problems
  - in math, one has to state explicitly all the assumptions that are being made

- But...
  - math alone cannot give us empirically valid theories
  - an economic model that is mathematically rigorous need not be relevant for our understanding of economic phenomena

Introduction

In this course, we will study four areas of mathematics that are heavily used in economics

- **linear algebra**
- **calculus of several variables**
- **optimization**
- **dynamical systems**

Linear algebra

What is it all about?

Example: What can we say about the existence of solutionf of the equation $ax = b$

- $x \in \mathbb{R}$ is an endogenous variable
- $a, b \in \mathbb{R}$ are exogenous

What if we had more linear equations?

How far can we get with just linear equations?

Systems of linear equations

The key object of study in linear algebra are *linear equations*

A linear equation has the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where

- $a_1, a_2, \ldots, a_n, b$ are fixed real numbers (parameters)
- $x_1, x_2, \ldots, x_n$ are the variables

An example from macroeconomics: IS-LM model

$$sY + ar = I^0 + G \tag{IS}$$

$$mY - hr = M_s - M^0 \tag{LM}$$

where:

- s is the marginal propensity to save
- Y is national income (GDP)
- r is the interest rate
- a and I^0 are parameters that determine investment
- G is public expenditure
- M^s is the money supply
- M^0, m and h are parameters that determine the demand for money

An example from macroeconomics: IS-LM model

- The IS-LM postulates that Y and r are the endogenous variables
- The aim of the model is to understand how the endogenous variables Y and r are determined for a given configuration of exogenous variables
- In other words, we look for a solution (Y, r) of the system of two linear equations (IS)-(LM)
- In addition, the model allows us to understand how a change in one of the exogenous variables, say G, affects the solution of the system

Method of substitution

Suppose we have a system of two linear equations in two unknowns:

$$2x_1 + 3x_2 = 7 \tag{1}$$

$$2x_1 + x_2 = 4 \tag{2}$$

How to solve it?

You may already be familiar with the method of **substitution**:

1. Rewrite (2) as $x_2 = 4 - 2x_1$
2. Substitute $x_2 = 4 - 2x_1$ into (1):

$$2x_1 + 3(4 - 2x_1) = 7 \tag{3}$$

3. Solve (3) for x_1, which gives $x_1 = \frac{5}{4}$
4. Finally, substitute $x_1 = \frac{5}{4}$ into $x_2 = 4 - 2x_1$ so as to obtain $x_2 = \frac{3}{2}$
5. Thus the solution of our system is $x_1 = \frac{5}{4}$ and $x_2 = \frac{3}{2}$

Method of substitution

Given a system of linear equations, you can always try and solve it by substitution (also known as direct elimination of variables)

However, the method of substitution:

- is not necessarily the best or fastest algorithm to find a solution
- does not provide us with a general theory of the existence and the multiplicity of solutions

More general method: Gauss-Jordan elimination

Example: two products

Demand $Q_i^d = K_i P_1^{\alpha_{i1}} P_2^{\alpha_{i2}} Y^{\beta_i}$, $i = 1, 2$

- P_i price of product i, Y income
- α_{ij} elasticity of demand of product i for price j
- β_i income elasticity of product i

Supply $Q_i^s = M_i P_i^{\gamma_i}$, $i = 1, 2$

- γ_i is the price elasticity of supply

Equilibrium $Q_i^d = Q_i^s$, $i = 1, 2$

- Notations $q_i^d = \ln Q_i^d$, $q_i^s = \ln Q_i^s$, $p_i = \ln P_i$, $y = \ln Y$, $m_i = \ln M_i$, $k_i = \ln K_i$

Example: two products

- By taking a logarithm of supply, demand, and the equilibrium condition we get a linear system

$$\begin{aligned} q_i^d &= k_i + \alpha_{ii} p_i + \alpha_{ij} p_j + \beta_i y, \\ q_i^s &= m_i + \gamma_i p_i, \\ q_i^s &= q_i^d. \end{aligned}$$

- after some algebraic manipulation this results to a system of the form

$$\begin{array}{ccc} (\alpha_{11} - \gamma_1) p_1 & + \alpha_{12} p_2 & = & m_1 - k_1 - \beta_1 y \\ \alpha_{21} p_1 & + (\alpha_{22} - \gamma_2) p_2 & = & m_2 - k_2 - \beta_2 y \end{array}$$

- Which of the variables are exogenous and endogenous?

Example: two products

- Eliminate p_1 from the first equation:

$$p_1 = \frac{m_1 - k_1 - \beta_1 y - \alpha_{12} p_2}{\alpha_{11} - \gamma_1}$$

- Substitute p_1 into the second equation

$$\begin{aligned}
p_2 &= \frac{m_2 - k_2 - \beta_2 y - \alpha_{21} p_1}{\alpha_{22} - \gamma_2} \\
&= \frac{m_2 - k_2 - \beta_2 y - \alpha_{21} \frac{m_1 - k_1 - \beta_1 y - \alpha_{12} p_2}{\alpha_{11} - \gamma_1}}{\alpha_{22} - \gamma_2} \\
&= \frac{(\alpha_{11} - \gamma_1)(m_2 - k_2 - \beta_2 y) - \alpha_{21}(m_1 - k_1 - \beta_1 y - \alpha_{12} p_2)}{(\alpha_{11} - \gamma_1)(\alpha_{22} - \gamma_2) - \alpha_{12}\alpha_{21}}.
\end{aligned}$$

- Substitute p_2 with the above expression in the formula of p_1, and we have a solution
- Is there something wrong with this approach?

Gauss-Jordan Elimination

Suppose we have a system of three linear equations in three unknowns:

$$\begin{aligned} x_1 - 0.4x_2 - 0.3x_3 &= 130 && \text{(a)} \\ -0.2x_1 + 0.88x_2 - 0.14x_3 &= 74 && \text{(b)} \\ -0.5x_1 - 0.2x_2 + 0.95x_3 &= 95 && \text{(c)} \end{aligned}$$

Multiply each side of (a) by 0.2 and then add it to (b). We obtain a new system, which is equivalent to the old one:

$$\begin{aligned} x_1 - 0.4x_2 - 0.3x_3 &= 130 && \text{(a)} \\ 0.8x_2 - 0.2x_3 &= 100 && \text{(b')} \\ -0.5x_1 - 0.2x_2 + 0.95x_3 &= 95 && \text{(c)} \end{aligned}$$

Gauss-Jordan Elimination

Similarly, multiply each side of (a) by 0.5 and then add it to (c):

$$\begin{aligned} x_1 - 0.4x_2 - 0.3x_3 &= 130 && \text{(a)} \\ 0.8x_2 - 0.2x_3 &= 100 && \text{(b')} \\ -0.4x_2 + 0.8x_3 &= 160 && \text{(c')} \end{aligned}$$

Multiply each side of (b') by 0.5 and then add it to (c'). We obtain a new system, which is equivalent to the old one:

$$\begin{aligned} x_1 - 0.4x_2 - 0.3x_3 &= 130 && \text{(a)} \\ 0.8x_2 - 0.2x_3 &= 100 && \text{(b')} \\ 0.7x_3 &= 210 && \text{(c'')} \end{aligned}$$

Note: now we could easily solve the system (a)-(b')-(c'') by substitution. The solution is

$$x_1 = 300, x_2 = 200, x_3 = 300$$

Gauss-Jordan Elimination

Instead of using substitution, suppose we continue our process of multiplying and adding equations. First of all, for each equation in (a)-(b')-(c''), multiply both sides so that the first non-zero coefficient is 1. We should obtain:

$$x_1 - 0.4x_2 - 0.3x_3 = 130 \qquad \text{(a)}$$

$$x_2 - 0.25x_3 = 125 \qquad \text{(b'')}$$

$$x_3 = 300 \qquad (c''')$$

Now multiply both sides of (c''') by 0.25 and then add it to (b''):

$$x_1 - 0.4x_2 - 0.3x_3 = 130 \qquad \text{(a)}$$

$$x_2 = 200 \qquad (b''')$$

$$x_3 = 300 \qquad (c''')$$

Gauss-Jordan Elimination

Finally, multiply ($c^{'''}$) by 0.3 and then add it to (a). In addition, multiply ($b^{'''}$) by 0.4 and then add it to (a) so as to get

$$x_1 = 300 \tag{a'}$$
$$x_2 = 200 \tag{$b^{'''}$}$$
$$x_3 = 300 \tag{$c^{'''}$}$$

Gauss-Jordan Elimination

A linear system of m equations in n unknowns is a list:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad\quad \vdots \qquad & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m,
\end{aligned}
$$

where

- $x_1, x_2, \ldots, x_n$ are the unknowns
- for every $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, a_{ij} and b_i are given real numbers (parameters)
- a_{ij} is the coefficient of the unknown x_j in the ith equation
- a solution is an n-tuple of real numbers $x_1, x_2, \ldots, x_n$ such that each of the m equations is satisfied

Gauss-Jordan Elimination

Given a linear system of m equations in n unknowns, the **Gauss-Jordan** elimination method consists in transforming the initial system into an equivalent, reduced system like (a')-(b''')-(c''') via a sequence of *elementary equation operations*:

- multiplying both sides of an equation by a nonzero real number
- adding a multiple of one equation to another
- interchanging (swapping) two equations

Existence and Multiplicity of Solutions

- Each of the two systems (1)-(2) and (a)-(b)-(c) has a solution, and that solution is unique
- In general, a system does not necessarily have a solution
  - Example:

$$\begin{aligned} x_1 + 4x_2 &= 2 \\ 2x_1 + 8x_2 &= 1 \end{aligned}$$

- In addition, if a system has a solution, the solution is not necessarily unique
  - Example:

$$\begin{aligned} x_1 + 4x_2 &= 2 \\ 2x_1 + 8x_2 &= 4 \end{aligned}$$

- In the next lectures, we will build on the Gauss-Jordan elimination method to develop of general theory of the existence and multiplicity of solutions for systems of linear equations

Exercise

Use the Gauss-Jordan elimination method to solve the following system of 4 linear equations in 4 unknowns:

$$\begin{aligned} 2x_1 - x_2 &= 0 \\ -x_1 + 2x_2 - x_3 &= 0 \\ -x_2 + 2x_3 - x_4 &= 0 \\ -x_3 + 2x_4 &= 5 \end{aligned}$$