ELEC-E8107 - Stochastic models, estimation and control

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Exercises Session 1

Exercise 1

Given the random variables x and y of dimensions n_x and n_y, with means $\bar{x}$ and $\bar{y}$ respectively, and with covariances matrices P_{xx}, P_{yy} and P_{xy}:

1. Find the mean and covariances of the n_z-dimensional vector $z = Ax + By + c$, where A and B are matrices of appropriate dimensions and c is a vector.

2. Indicate the dimensions of A, B and c.

Solution Exercise 1

1. The mean and covariance of z:
Since the expected value of the random variable is a linear operator and the expected value of a constant is a constant. The mean of the random variable $z = Ax + By + c$ is then computed as:

$$\begin{aligned}\bar{z} &= E[Ax + By + c] \\ &= E[Ax] + E[By] + E[c] \\ &= AE[x] + BE[y] + c \\ &= A\bar{x} + B\bar{y} + c\end{aligned}$$

Here, the mean $E[x] = \bar{x}$ is the result of an n-fold integration written as;

$$\begin{aligned}E[x] &= \int_{-\infty}^{\infty} \dots \int_{\infty}^{\infty} xp(x)dx_1 \dots dx_n \\ &= \bar{x}\end{aligned}$$

Similarly, the P_{xx} is the covariance matrix computed by the n_x-fold integration, given as:

$$\begin{aligned} cov(x,x) &= E[(x-\bar{x})(x-\bar{x})^T] \\ &= \int_{-\infty}^{\infty} \dots \int_{\infty}^{\infty} (x-\bar{x})(x-\bar{x})^T p(x) dx_1 \dots dx_n \\ &= P_{xx} \end{aligned}$$

Thus, the covariance matrix P_{zz} of the n_z-vector is obtained from the n_z-fold integration (here, we will use the short-hand notations), which leads to;

$$\begin{aligned} P_{zz} &= E[(z-\bar{z})(z-\bar{z})^T] \\ &= E\Big[(Ax+By+c-(A\bar{x}+B\bar{y}+c))(Ax+By+c-(A\bar{x}+B\bar{y}+c)^T\Big] \\ &= E\Big[\,(A(x-\bar{x})+B(y-\bar{y})+c(1-1))\,(A(x-\bar{x})+B(y-\bar{y})+c(1-1))^T\,\Big] \\ &= E\Big[\,(A(x-\bar{x})+B(y-\bar{y}))\,(A(x-\bar{x})+B(y-\bar{y}))^T\,\Big] \\ &= E\Big[A(x-\bar{x})(x-\bar{x})^T A^T + A(x-\bar{x})(y-\bar{y})^T B^T + \\ &\qquad B(y-\bar{y})(x-\bar{x})^T A^T + B(y-\bar{y})(y-\bar{y})^T B^T\Big] \\ &= E\left[A(x-\bar{x})(x-\bar{x})^T A^T\right] + E\left[A(x-\bar{x})(y-\bar{y})^T B^T\right] \\ &\qquad E\left[B(y-\bar{y})(x-\bar{x})^T A^T\right] + E\left[B(y-\bar{y})(y-\bar{y})^T B^T\right] \\ &= AE\left[(x-\bar{x})(x-\bar{x})^T\right] A^T + AE\left[(x-\bar{x})(y-\bar{y})^T\right] B^T \\ &\qquad BE\left[(y-\bar{y})(x-\bar{x})^T\right] A^T + BE\left[(y-\bar{y})(y-\bar{y})^T\right] B^T \\ &= AP_{xx}A^T + AP_{xy}B^T + BP_{yx}A^T + BP_{yy}B^T \end{aligned}$$

2. The dimensions of A, B and c:

The dimensions of matrix A is $n_z \times n_x$; of matrix B is $n_z \times n_y$; and, the dimension of vector c is $n_z \times 1$.

Note that the covariance matrices are the symmetric matrices, where the diagonal entries constitute the variances and off-diagonal elements contain (scalar) covariances. The covariance matrix is also positive definite (and thus non-singular) unless there is some dependence among the elements of vector random variable. In such cases, the covariance matrix is positive semi-definite.

Exercise 2

A nonlinear system dynamic model of the car shown in Fig 1 is given by the following equation.

$$\begin{cases} x_{k+1} = x_k + cos(\theta_k)\Delta t_k v_k \\ y_{k+1} = y_k + sin(\theta_k)\Delta t_k v_k \\ \theta_{k+1} = \theta_k \frac{\Delta t_k v_k}{L} tan(\Phi_k) \\ v_{k+1} = v_k \\ \Phi_{k+1} = \Phi_k \end{cases} \tag{1}$$

Where v is the speed of the vehicle, θ is the heading and Φ the steering angle. The state of the vehicle can be define as the vector $X_k = \left[x_k, y_k, \theta_k, v_k, \Phi_k\right]^T$. The equation (1) can be written as $X_{k+1} = f(X_k, t_k)$.

1. Compute the Jacobian of the function $f(X_k, t_k)$ with respect to the state vector X_k

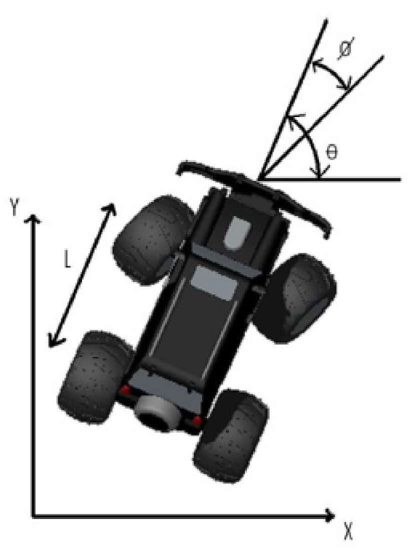

Figure 1: Simple car kinematic model markings: The distance between the axles of the vehicle is described by L, the direction is described by θ and the steering angle by Φ. The vehicle navigation point is in the center of the rear axle.

Solution Exercise 2

The Jacobian is computed using the definition. Note that $f(X_k, t_k)$ is a vector valued function. The Jacobian can be written as follow:

$$\frac{\partial f}{\partial X}\bigg|_{X=X_k} = \begin{bmatrix} \frac{\partial f_1}{\partial X_1} & \frac{\partial f_1}{\partial X_2} & \frac{\partial f_1}{\partial X_3} & \frac{\partial f_1}{\partial X_4} & \frac{\partial f_1}{\partial X_5} \\ \frac{\partial f_2}{\partial X_1} & \frac{\partial f_2}{\partial X_2} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} 1 & 0 & -v_k \Delta t_k sin(\theta_k) & \Delta t_k cos(\theta_k) & 0 \\ 0 & 1 & v_k \Delta t_k cos(\theta_k) & \Delta t_k sin(\theta_k) & 0 \\ 0 & 0 & 1 & \Delta t_k \frac{tan(\Phi_k)}{L} & \frac{v_k \Delta t_k}{L cos(\Phi)^2} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

Exercise 3

Prove that the following equation holds for a discrete time Markov process

$$\int p(x_k|x_{k-1})p(x_{k-1}|x_{k-2})dx_{k-1} = p(x_k|x_{k-2})$$

Solution Exercise 3

The marginal joint distribution $p(x_k, x_{k-2})$ can be obtained from the joint distribution $p(x_k, x_{k-1}, x_{k-2})$ by integrating with respect to x_{k-1}:

$$\int p(x_k, x_{k-1}, x_{k-2})dx_{k-1} = p(x_k, x_{k-2}) \tag{4}$$

$$= p(x_k|x_{k-2})p(x_{k-2}) \tag{5}$$

By recursively using the Bayes' formula, and the Markov property we can write the following:

$$p(x_k, x_{k-1}, x_{k-2}) = p(x_k|x_{k-1}, x_{k-2})p(x_{k-1}, x_{k-2}) \tag{6}$$

$$= p(x_k|x_{k-1})p(x_{k-1}, x_{k-2}) \tag{7}$$

$$= p(x_k|x_{k-1})p(x_{k-1}|x_{k-2})p(x_{k-2}) \tag{8}$$

The equation (5) is obtained from (4) by applying the Markov property. We integrate (6) and equate it with (3):

$$\int p(x_k|x_{k-1})p(x_{k-1}|x_{k-2})p(x_{k-2})dx_{k-1} = p(x_k|x_{k-2})p(x_{k-2}) \tag{9}$$

$$p(x_{k-2})\int p(x_k|x_{k-1})p(x_{k-1}|x_{k-2})dx_{k-1} = p(x_k|x_{k-2})p(x_{k-2}) \tag{10}$$

$$\int p(x_k|x_{k-1})p(x_{k-1}|x_{k-2})dx_{k-1} = p(x_k|x_{k-2}) \tag{11}$$

Exercise 4

The covariance matrix of random variables X and Y happens to be:

$$Q = \begin{bmatrix} 4 & -3 \\ -3 & 9 \end{bmatrix}$$

1. Find the variance of X and Y?
2. Compute the correlation coefficient between the two random variables.

Solution Exercise 4

1. The variance of X and Y:
The diagonal values of a covariance matrix consists of variances of individual variables, thus $Var(X) = 4$, and $Var(Y) = 9$.
2. The correlation coefficient between X and Y:
The off-diagonal elements include information of the correlation coefficient,

$$\begin{aligned} Q_{XY} &= Q_{YX} \\ &= \sigma_X * \sigma_Y * \rho_{X,Y} \end{aligned}$$

Thus, $\rho_{X,Y} = \frac{-3}{2*3} = -0.5$. Notice that the value of a correlation coefficient (also know as Pearson's correlation coefficient) is always between +1 and -1.
Note: if $\rho_{X,Y} = 0$, the variables X and Y are said to be uncorrelated. If $\rho_{X,Y} = 1$, then the variables X and Y are said to be linearly dependent.