Aalto University
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MS-E2114 Investment Science
Lecture 2: Fixed income securities

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Overview

This lecture

- **Fixed income securities** are debt instruments which provide (i) returns in the form of regular, or fixed, interest payments and (ii) repayments of the principal when the security reaches maturity
- Refers especially to bonds that pay a periodic, fixed, non-random coupon and other similar instruments
  - E.g., bonds issued by corporations, sovereign states, municipalities, etc.
- The term also encompasses instruments whose interest depends on a reference rate such as EURIBOR
- In this lecture, we derive valuation formulas for several types of fixed income securities

Overview

Terminology

- **Financial instruments** are monetary contracts between parties
  - Cash, evidence of ownership, contract to receive cash or other financial instruments
- **Security** = A readily transferable financial instrument which is traded in a well-developed market
  - Governed by laws on public market securities (e.g. regarding insider information)
- **Fixed income instrument** = A financial instrument which provides a preagreed stream of cash flows
  - Interest rate may be fixed or depend on a possibly changing reference rate such as EURIBOR
  - Yet the issuer may **default** (e.g., bankruptcy)
- **Fixed income security** = A fixed income instrument which is a security, too

Examples of fixed income instruments

- Money market instruments (securities)
  - Tradeable instruments that offer a guaranteed interest (possibly floating-rate) for a short period of time
  - Certificates of deposit offered by banks
  - Short term notes (1 yr or less) by corporations (= commercial paper)
- U.S. government securities
  - Treasury bills ("T-bills") < 1 year
  - Treasury notes 2-10 years
  - Treasury bonds > 10 years
- Other bonds (securities)
  - Municipal bonds (issued by e.g. by cities)
  - Corporate bonds

Further examples (these are not usually securities)

- **Mortgages** are used by purchasers of real property to raise funds to buy real estate, or alternatively by existing property owners to raise funds for any purpose, while putting a lien on the property being mortgaged
  - **Reverse mortgages** allow owners of home equity to borrow against the value of their home
- **Annuities** provide a series of payments at equal intervals (e.g., regular deposits to a savings account, monthly insurance payments, and pension payment)

Credit ratings

- Bonds offer a **fixed cash flow** *unless* the issuer defaults
- Ratings provided by credit rating agencies
  - The 'Big Three': Moody's, Standard & Poor's, Fitch
- U.S. Treasury securities have been considered the least risky historically
- Ratings are traditionally grouped into grades of *Investment Grade* (IG) and *High Yield* (HY)
  - Investment grade = Baa3 or better (Moody's), BBB- or better (S&P, Fitch)
  - High yield = Ba1 or worse (Moody's), BB+ or worse (S&P, Fitch)
  - High yield bonds are also known as *junk bonds*
  - `https://en.wikipedia.org/wiki/Credit_rating`

Market for future cash

- Fixed income securities define the time value of money
- Markets for many types of these securities are extremely well-developed
- Market prices for these securities reflect
  1. Time value of money
  2. Fundamental risk premium (= compensation for the risk that the issuer defaults)
  3. Market price risk premium (= compensation for the risk that the value of the security goes down)
  4. Supply and demand

Overview

Perpetual annuity

- Receive A € annually forever starting in a year's time
  - r = interest rate
- The net present value of this cash flow stream is ($r > 0$)

$$
\begin{aligned}
P &= \sum_{k=1}^{\infty} \frac{A}{(1+r)^k} = \frac{A}{1+r} + \frac{1}{1+r} \sum_{k=1}^{\infty} \frac{A}{(1+r)^k} \\
&= \frac{A}{1+r} + \frac{P}{1+r} \\
\Rightarrow P &= \frac{A}{r}
\end{aligned}
$$

- If $A = 25000$ € and $r = 0.1 = 10\%$, then
  $P = 25\,000$ € /0.1= 250 000 €

Cash flows of finite length

- Get A € annually for the next n years
  - Future cash flows discounted with interest rate r
- Present value can be computed from the perpetual annuity by substracting the cash flows that occur after the n-th year

$$
\begin{aligned}
P &= \sum_{k=1}^{n} \frac{A}{(1+r)^k} \\
&= \underbrace{\sum_{k=1}^{\infty} \frac{A}{(1+r)^k}}_{\frac{A}{r}} - \underbrace{\sum_{k=n+1}^{\infty} \frac{A}{(1+r)^k}}_{\frac{1}{(1+r)^n}\sum_{k=1}^{\infty}\frac{A}{(1+r)^k} = \frac{1}{(1+r)^n}\frac{A}{r}}
\end{aligned}
$$

$$
\Rightarrow P = \frac{A}{r}\left(1 - \frac{1}{(1+r)^n}\right)
$$

Examples of streams with finite life

- Get $A = 25\,000$ € annually for 20 years
  - Interest rate $r = 0.1 = 10\%$

$$P = \frac{25\,000\,€}{0.1}\left(1 - \frac{1}{(1+0.1)^{20}}\right) = 213\,000\,€$$

  - PV "only" 37 000 € less than that of the corresponding perpetual annuity on slide 11
  - For high interest rates, the cash flows that occur in the more distant future do not matter that much

Example: Consumer credit

- Loan $P = 10\,000€$
  - Interest is paid monthly at the nominal rate of 12% p.a. $\Rightarrow r = 0.12/12 = 0.01$
  - Amortize (=pay back) in 3 years in **equal monthly payments** ($n = 36$)
- What are the monthly payments A?
  - Finite life stream with unknown A
- Solving the pricing equation on slide 11 for A gives

$$A = \frac{r(1+r)^n P}{(1+r)^n - 1} = \frac{0.01(1+0.01)^{36} 10\,000\,€}{(1+0.01)^{36} - 1} \approx 332.14€$$

- Sum of payments $36 \times 332.14\,€ = 11\,957.04\,€$
- Amortization table
  - (Loan at $n+1$) = (Loan at n) – (Amortization at n)

Example: Consumer credit

n	Loan capital	Interest	Amort.	Payment		n	Loan ca.	Inter.	Amort.	Paym.
0	10000.0	100.0	232.1	332.1						
1	9767.9	97.7	234.5	332.1		19	5169.0	51.7	280.5	332.1
2	9533.4	95.3	236.8	332.1		20	4888.5	48.9	283.3	332.1
3	9296.6	93.0	239.2	332.1		21	4605.3	46.1	286.1	332.1
4	9057.4	90.6	241.6	332.1		22	4319.2	43.2	289.0	332.1
5	8815.9	88.2	244.0	332.1		23	4030.2	40.3	291.8	332.1
6	8571.9	85.7	246.4	332.1		24	3738.4	37.4	294.8	332.1
7	8325.5	83.3	248.9	332.1		25	3443.6	34.4	297.7	332.1
8	8076.6	80.8	251.4	332.1		26	3145.9	31.5	300.7	332.1
9	7825.2	78.3	253.9	332.1		27	2845.2	28.5	303.7	332.1
10	7571.3	75.7	256.4	332.1		28	2541.6	25.4	306.7	332.1
11	7314.9	73.2	259.0	332.1		29	2234.8	22.4	309.8	332.1
12	7055.9	70.6	261.6	332.1		30	1925.0	19.3	312.9	332.1
13	6794.3	67.9	264.2	332.1		31	1612.2	16.1	316.0	332.1
14	6530.1	65.3	266.8	332.1		32	1296.1	13.0	319.2	332.1
15	6263.3	62.6	269.5	332.1		33	977.0	9.8	322.4	332.1
16	5993.8	59.9	272.2	332.1		34	654.6	6.6	325.6	332.1
17	5721.6	57.2	274.9	332.1		35	329.0	3.3	328.9	332.1
18	5446.6	54.5	277.7	332.1		36	0.1	≈ 0		

Overview

Introduction

Annuities

Bonds

Duration

Immunization

Links

Bonds as examples of fixed income securities

- A bond pays its **principal** (also known as **face value** or **par value**) on its **maturity date**
  - E.g. pay a face value of 1000 € on 1.1.2029
- Most bonds pay periodic interest payments (also known as **coupon payments**)
  - **Coupon rate** defined as a percentage of face value
    - E.g., coupon rate 9 % p.a. $\Rightarrow$ receive 90 € on every 1st of January until 1.1.2029
  - Coupon rate of a par bond (issue price = face value) is close to the prevailing interest rate when the bond is issued
  - If the bond is issued at a discount (issue price is less than face value), then the coupon rate can be less
    - Price appreciation to maturity generates further return to the investor
  - Originally, actual coupons were attached to printed bond certificates

Bond yield

- **Yield (To Maturity)(YTM)** = Internal rate of return (IRR) of a bond per annum (p.a.) at its current price
  (i) Face value F
  (ii) m coupon payments p.a., each payment C/m
  (iii) n periods (payments) left to maturity
- If the current price is $P \Rightarrow$, then YTM is the rate λ such that

$$P = \frac{F}{[1+(\lambda/m)]^n} + \underbrace{\sum_{k=1}^{n} \frac{C/m}{[1+(\lambda/m)]^k}}_{\text{Finite life stream}}$$

$$\Rightarrow P = \frac{F}{[1+(\lambda/m)]^n} + \frac{C}{\lambda}\left(1 - \frac{1}{[1+(\lambda/m)]^n}\right)$$

The formula is best understood if you substitute $r = \lambda/m$ as the periodic (e.g. monthly) IRR, and then obtain an annualized rate (= YTM) by linear annualization $\lambda = rm$.

Bond yield

- As with IRR in general, YTM (λ) is computed numerically
- **Accrued interest** (AI) tells how much interest the bond has accrued since the last coupon payment
  - Linear interpolation:

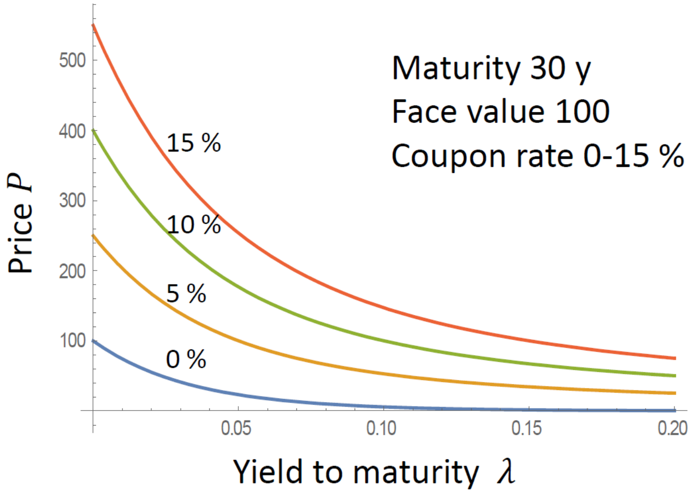
$$AI = \frac{\text{Days since last coupon}}{\text{Days in period}} \times \text{Coupon payment}$$

  - Consider a bond which has face value 1 000 €, coupon rate 9 % with coupon payments every Feb 15 and Aug 15.
  - If this bond is bought on May 5, there are 83 days since last coupon payment and 99 days until next payment
  - $AI = \frac{83}{83+99} \times \frac{9\%}{2} \times 1\,000€ = 20.52€$

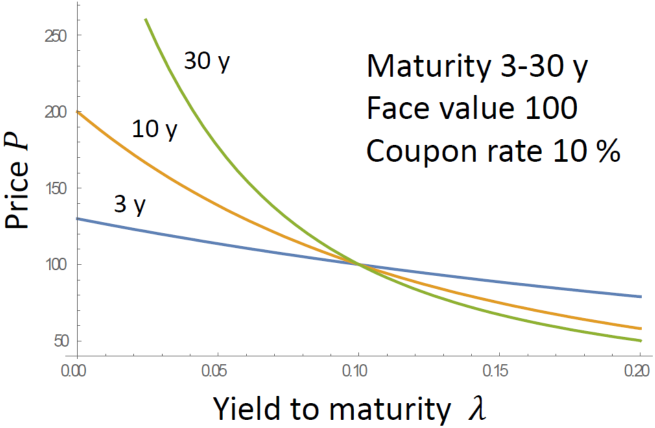
Example of bond quotes

U.S. Treasury Quotes Wednesday, January 13, 2016					
Treasury Notes & Bonds					
Maturity	**Coupon**	**Bid**	**Asked**	**Chg**	**Asked yield**
1/31/2020	1.250	99.45	99.47	0.0469	1.385
1/31/2020	1.375	99.93	99.95	0.0234	1.389
2/15/2020	3.625	108.82	108.84	0.0391	1.393
2/15/2020	8.500	128.23	128.25	0.0547	1.369
2/29/2020	1.250	99.38	99.39	0.0703	1.402
2/29/2020	1.375	99.91	99.93	0.0938	1.393
3/31/2020	1.125	98.84	98.85	0.1250	1.407
3/31/2020	1.375	99.82	99.84	0.0859	1.415
4/30/2020	1.125	98.80	98.81	0.1094	1.411
4/30/2020	1.375	99.76	99.77	0.0703	1.429
5/15/2020	3.500	108.56	108.58	0.0703	1.451
5/15/2020	8.750	130.61	130.63	0.1094	1.437
5/31/2020	1.375	99.66	99.68	0.1172	1.451
Coupon: Annual rate (%)					
Ask and bid prices: % of face value					
Chg: Daily change in asked price					
Asked yield: yield to maturity at asked price					

Price-yield curve



Price-yield curve



Price-yield curve

- Yield $\lambda = 0$ if and only if the price equals the total cash flow, that is, $P = F + nC/m$
- Yield λ equals the coupon rate if and only if price P = face value F
- If the yield λ increases, the price P of bonds with small coupon rates decline more than the price of bonds with high coupon rates
  - The bigger early coupon payments are less affected by the rising interest rates
- For bonds of longer maturity, the price-yield curve is steeper
  - Steeper curve $\iff$ bond prices are more sensitive to the interest rate

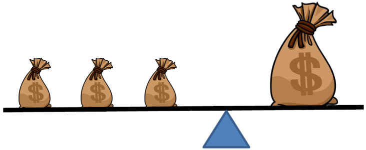
Overview

Duration

- Consider the cash flow stream $(c_0, c_1, \ldots, c_n)$ which gives c_i at time $t_i, i = 0, \ldots, n$
- The **duration** of this cash flow stream is

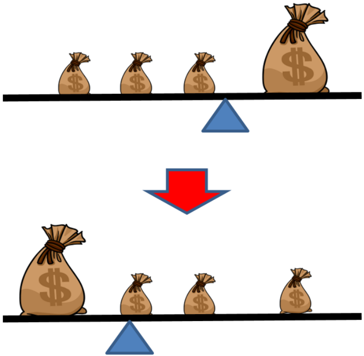
$$D = \frac{PV(t_0)t_0 + PV(t_1)t_1 + \cdots + PV(t_n)t_n}{PV},$$

  where $PV(t_i), i = 0, 1, \ldots, n$ is the present value of c_i and $PV = PV(t_0) + \cdots + PV(t_n)$
- This is the PV weighted average of the payment times $t_0, t_1, \ldots, t_n$ of the cash flow stream $(c_0, c_1, \ldots, c_n)$
  - i-th weight = share of the $PV(t_i)$ out of the total PV
  - By definition, these weights add up to one



Duration

- Duration is a measure of the bond's 'average' maturity
  - If there are no coupons, then duration = maturity
  - If there are coupons, then duration < maturity
  - For two bonds with the same total cash flow (i.e., coupons + face value), the duration is shorter for the one with higher coupon rate



Macaulay duration

- What interest rate r should one use when computing duration?
- Macaulay duration: r = YTM

$$\Rightarrow \quad D = \frac{\sum_{k=1}^{n} \frac{k}{m} \frac{c_k}{(1+\frac{\lambda}{m})^k}}{PV}, \text{ where}$$

$$PV = \sum_{k=1}^{n} \frac{c_k}{(1+\frac{\lambda}{m})^k}$$

- For bonds (derived in exercises)

$$D = \frac{1+y}{my} - \frac{1+y+n(c-y)}{mc[(1+y)^n - 1] + my}, \text{ where}$$

m = periods per year　　n = nr of periods left

$y = \lambda/m$ = yield per period　　c = coupon rate

Modified duration

- The present value of cash flow c_k is

$$PV_k = \frac{c_k}{[1+(\lambda/m)]^k}$$

$$\Rightarrow \frac{dPV_k}{d\lambda} = -\frac{(k/m)c_k}{[1+(\lambda/m)]^{k+1}} = -\frac{(k/m)PV_k}{1+(\lambda/m)}$$

- The price sensitivity of a bond is

$$P = \sum_{k=1}^{n} PV_k$$

$$\Rightarrow \frac{dP}{d\lambda} = \sum_{k=1}^{n} -\frac{(k/m)PV_k}{1+(\lambda/m)} = -\frac{1}{1+(\lambda/m)} \frac{\sum_{k=1}^{n}(k/m)PV_k}{P} P$$

$$\Rightarrow \frac{dP}{d\lambda} = -\frac{1}{1+(\lambda/m)} DP = -D_M P,$$

where D_M is the **modified duration** $D_M = D/[1+(\lambda/m)]$

Applying modified duration

- Consider a bond such that
  - Maturity 30 y, no coupons (i.e., coupon rate 0 %)
- Assume that interest rates rise from 10 % to 11%

$$\lambda \to \lambda + \Delta\lambda, \quad \lambda = 0.1, \Delta\lambda = 0.01$$

- No coupons $\Rightarrow D =$ Maturity $\Rightarrow D_M = 30/[1+0.1] \approx 27.27$
- Linear approximation:

$$\Delta P \approx -D_M P \Delta\lambda$$
$$\Rightarrow \frac{\Delta P}{P} \approx -D_M \Delta\lambda = -27.27 \times 0.01 = -27.27\%$$

  - Price approx. sinks by 27 % if the interest rate rises by 1%

The actual price drop when yield changes from 10% to 11% is 23.78%. For smaller changes the approximation is more accurate: For 0.1% change, the drop is 2.689% and for 0.01% change, 0.2723%.

Application of modified duration

- Consider another bond with
  - Maturity 30 y, coupon rate 10 %, 2 coupons per year
  - Price = face value, i.e., YTM is 10 %
- Macaulay duration
  $D = 9.938 \Rightarrow D_M = 9.938/[1 + (0.1/2)] \approx 9.47$
- Price change

$$\frac{\Delta P}{P} \approx -D_M \Delta\lambda = -9.47 \times 0.01 = -9.47\%$$

  - The relatively decline in price is now much less because of the coupon payments

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The actual price drop when yield changes from 10% to 11% is 8.72%. For smaller changes the approximation is more accurate: For 0.1% change, the drop is 0.9386% and for 0.01% change, 0.09457%.

Duration of a portfolio

- **Portfolio** of bonds = a collection consisting of a set of m bonds
- Price $P = P_1 + P_2 + \cdots P_m$
  - $P_i =$ price of bond $i = 1, 2, \ldots, m$

Theorem

*(**Duration of a portfolio**) Suppose there are m fixed-income securities with prices and durations of P_i and D_i, respectively, $i = 1, 2, \ldots, m$, all computed using the same yield. Then the price P and duration D of the portfolio consisting of the aggregate of these securities are*

$$P = P_1 + P_2 + \cdots + P_m$$
$$D = w_1 D_1 + w_2 D_2 + \cdots + w_m D_m,$$

where $w_i = P_i / P, i = 1, 2, \ldots, m$

Duration of a portfolio

- **Proof.** Outline for the case of two securities A and B:

$$D^{A+B} = \sum_{k=0}^{n} \frac{PV_k^{A+B} t_k}{P^{A+B}}$$

$$D^{A} = \sum_{k=0}^{n} \frac{PV_k^{A} t_k}{P^{A}}$$

$$D^{B} = \sum_{k=0}^{n} \frac{PV_k^{B} t_k}{P^{B}}$$

$$\Rightarrow P^A D^A + P^B D^B = \sum_{k=0}^{n} t_k \left(PV_k^A + PV_k^B \right)$$

Duration of a portfolio

- Divide both sides of equation by $P = P^{A+B} = P^A + P^B$

$$\Rightarrow \frac{P^A D^A + P^B D^B}{P} = \frac{\sum_{k=0}^{n} t_k \left(PV_k^A + PV_k^B \right)}{P} = D$$

$$\Rightarrow D = \frac{P^A}{P} D^A + \frac{P^B}{P} D^B$$

- By definition, duration of a portfolio of A and B is

$$D^{A+B} = \sum_{k=0}^{n} \frac{t_k PV_k^{A+B}}{P^{A+B}} = \sum_{k=0}^{n} \frac{t_k PV_k^{A+B}}{P}$$

- Thus, $PV_k^{A+B} = PV_k^A + PV_k^B$ implies $D^{A+B} = D$
  - Holds when payments from A and B are discounted <u>with the same rate</u> for each period k, assuming the same yield
  - This assumption of identical interest rates does not hold for Macaulay duration which uses YTM for each bond

Overview

Immunization

- **Immunization** is the development of an investment strategy when:
  - An investor has a liability stream (a cash flow stream the investor has to pay) that is sensitive to interest rates
  - He or she wants to construct an investment portfolio to match this liability stream both in terms of
    - present value and
    - interest rate sensitivity
- The combination of the portfolio and the liability stream is insensitive (immune) to (small) interest rate changes
- If an investor wants to immunize a bond portfolio, then he or she may have to short other bonds, which may be difficult or expensive in practice

Immunization

- **Principle**: Buy a portfolio of equal NPV whose interest rate sensitivity is the same as that of the liability stream being immunized
- If there are zero coupon bonds with many enough maturities, then perfect matching of cash flows is possible
  - Would match interest rate sensitivities exactly
  - This is difficult, however, because zero coupon bonds are rare and there may be no bonds whose maturities coincide with those of the cash flows of the portfolio
- The other method is to use **duration**
  - First-order (i.e., first derivative) approximation of interest rate sensitivity

Immunization

- **Task**: Immunize a liability stream with duration D and price P
- Bonds A and B available for immunization
- Buy A and B for total amount V_A and V_B (unit price times units bought) such that

$$
\begin{aligned}
P &= V_A + V_B \\
D &= w_A D_A + w_B D_B, \text{ where} \\
w_i &= \frac{V_i}{P}, i = A, B
\end{aligned}
$$

- In practice, more than two bonds would used
  - Helps diversify risk (of default)
  - Leads to more variables than equations $\Rightarrow$ there can be many solutions

Example: Immunization

- A company is liable to pay 1 million € in 10 years
  - No coupons ⇒ Duration 10 y
- Immunize using the following three bonds whose face value is 100 € and which pay two coupons per year

Bond	Coupon rate	Maturity (y)	Price (€)	YTM	Duration (y)
1	6%	30	69	9%	11.44
2	11%	10	113	9%	6.54
3	9%	20	100	9%	9.61

- The PV of the liability at the prevailing rate is

$$P = \frac{1\ 000\ 000€}{[1 + (0.09/2)]^{20}} \approx 414\ 634€$$

Example: Immunization

- If we use bonds 1 & 2

$$\begin{cases} P & = V_1 + V_2 \\ D & = \frac{V_1}{P} D_1 + \frac{V_2}{P} D_2 = 10 \end{cases}$$

$$\Rightarrow \begin{cases} V_1 & = P \frac{D - D_1}{D_1 - D_2} \approx 292\ 788 \\ V_2 & = P \frac{D_1 - D}{D_1 - D_2} \approx 121\ 854 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{V_1}{P_1} & = \frac{292\ 788}{69} = \textbf{4241 units of bond \# 1} \\ \frac{V_2}{P_2} & = \frac{121\ 854}{113} = \textbf{1078 units of bond \# 2} \end{cases}$$

Example: Immunization

- If we use bonds 2 & 3
  - No solution with positive amounts of bonds (weighted average of $D_2 = 6.54$ and $D_3 = 9.61$ less than $D = 10$ with all positive weights)

$$\begin{cases} V_2 & = P\frac{D_3 - D}{D_3 - D_2} \approx -52\,575 \\ V_3 & = P - V_2 \approx 467\,317 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{V_2}{P_2} & = -\frac{52\,575}{113} = \textbf{sell short 465 units of bond \# 2} \\ \frac{V_3}{P_3} & = \frac{467\,317}{100} = \textbf{purchase 4673 units of bond \# 3} \end{cases}$$

Example: Immunization

		Percent yield		
		9.0	8.0	10.0
Bond 1	Price	69.04	77.38	62.14
	Shares	4241.00	4241.00	4241.00
	Value	292798.64	328168.58	263535.74
Bond 2	Price	113.01	120.39	106.23
	Shares	1078.00	1078.00	1078.00
	Value	121824.78	129780.42	114515.94
Obligation	Value	414642.86	456386.95	376889.48
Surplus		-19.44	1562.05	1162.20

Overview

Introduction

Annuities

Bonds

Duration

Immunization

Links

Links

- Bloomberg bond rates
- Statistics on the Finnish central government debt
- Information on credit ratings
- Credit ratings of Finland
- Euro area yield curves
- Russia government bonds
- 10-year government bond spreads
- Debt structure of Stora Enso
- S&P credit rating of Stora Enso
- List of sovereign debt crises
- List of stock market crashes and bear markets

Overview

Introduction

Annuities

Bonds

Duration

Immunization

Links